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Abstract: Under the vertex representation for the state partitions, this paper studies the fragility of piecewise affine (PWA) control laws for constrained discrete-time linear systems. In particular, the aim is to evaluate the impact of uncertainty in the representation of state-space partition. Practically, the notion of *coupled vertex sensitivity* (CVS) is defined to inscribe the joint uncertainty in the vertex positioning. This novel concept reduces the computational complexity and the conservativeness of the construction compared to the use of independent vertex perturbation through the iterative procedure using the existing results. A numerical example illustrated the effectiveness of our work.

Keywords: Fragility; PWA control; Coupled vertex sensitivity.

1. INTRODUCTION

Model predictive control (MPC) is an effective and widely accepted method for complex constrained multivariable control problems in the process industries (Maciejowski, 2002). Based on the formulation of the optimal solution underlying the control law, MPC can be divided into two main categories: implicit MPC and explicit MPC (EMPC) (Rawlings et al. (2017)). The former approach iterates an online optimization process to compute a control action for a constrained control system based on current states and the strategy applied successfully to chemical processes on a large scale. However, the characteristics of online optimization limit its application to systems with high requirements from a computational perspective (small sampling time). EMPC, on the other hand, solves the problem to obtain an explicit control law with system states as parameters. This approach reduces the computational efforts of solving an online multi-parametric programming problem to a simple evaluation of a control law that is stored in the form of a look-up table (Bemporad et al. (2002)) and moves the computational problem to the storing and evaluation mechanisms.

The robustness of closed loop control systems requires uncertainty handling related to the design and implementation. In recent years, many researchers have investigated the robustness or fragility of EMPC from different perspectives. Given a polytopic uncertainty on the nominal model of the system, in Olaru et al. (2013), authors proposed an explicit robustness margin for discrete-time linear systems with PWA control. In Nguyen et al. (2016), authors extended this work to the direction of admissible variations in the PWA control law coefficients and denoted it as the fragility margin such that the positive invariance of systems states set is guaranteed. Different from the theoretical analysis of EMPC in the design process, the robustness analysis of the system model and the corresponding PWA controller, in Suardi et al. (2014), researchers investigated the impact of the unavoidable errors on the system performance during the practical application of EMPC. This work analyzed the quantization operation for the point of view of the state-space partitions and PWA control laws assuming that the states regions after quantizing are non-overlapping. In Koduri et al. (2017b), the conservative assumption of non-overlapping property was relaxed in the presence of quantization and the vertex sensitivity set satisfying the non-overlapping partition is given by theoretical analysis under the state partitions vertex representation. In Koduri et al. (2017a), authors showed that the non-overlapping and the invariance properties of a PWA controller are preserved when the perturbation takes place on the vertices.

The present paper pursues the analysis of the robustness of PWA control systems under the vertex representation of state partitions. With respect to the results mentioned above and available in the literature, the main contribution is related to the coupled vertex sensitivity (CVS), which is proposed as a concept to describe the simultaneous uncertainty in the location of two vertices of the original explicit PWA solution of the parametric optimization within MPC. Multi-vertex uncertainty is a complex phenomenon in the representation of the state-partitioned controllers and brings the robustness analysis to a relevant practical dimension. Indeed, previous work addressed only single-vertex uncertainty and treated the presence of uncertainty in multiple vertices in terms of sequential single vertex analysis. This approach has the shortcomings of computational complexity, high conservativeness in practice, and dependence of the solution on the processing order. By the proposed CVS, we partially cope with the shortcomings of sequential analysis and open the way for the characterization of uncertainty in terms of the half-space representation of the state partitions of the PWA controllers.

This paper is organised as follows. In the preliminaries, the background of EMPC and related research are presented. In section 3, the CVS and associated constructive approach are given. Then, an algorithm to calculate CVS is designed. A numerical example is presented to illustrate the effectiveness of the proposed CVS.

Notions: $\mathbb{R}$ and $\mathbb{N}$ denote the field of real numbers and the non-negative integers. $\mathcal{I}_N$ denotes the set $\{1, 2, \dots, N\}$, while $\mathbb{I}_N$ represents the N-dimensional identity matrix. The convex hull of a set of points $\{v_1, v_2, \dots, v_n\} \in \mathbb{R}^n$ is denoted $conv\{v_1, v_2, \dots, v_n\}$. For a given bounded polyhedral set $\mathcal{P} \subset \mathbb{R}^n$ the (finite) set of vertices is denoted $\mathcal{V}(\mathcal{P})$.

2. PRELIMINARIES

2.1 PWA control

Consider a discrete-time linear system:

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$ are the state vector and control input vector respectively at time k. The states and inputs are bounded by polytopes as:

$$x(k) \in \mathcal{X}, u(k) \in \mathcal{U}, \forall k \in \mathbb{N},$$
$$\mathcal{X} = \{x : H_x x \le b_x\}, \mathcal{U} = \{u : H_u u \le b_u\},$$

where b_x, b_u, H_x, and H_u are constant matrices with suitable dimension. Before referring to the PWA control problem of (1), the related definitions are recalled.

Definition 1. *A collection of polyhedral sets $\{\mathcal{X}_i\}_{\forall i \in \mathcal{I}_N}$ is called a polyhedral partition of $\mathcal{X}$ if*

$$\mathcal{X} = \cup_{i=1}^{N} \mathcal{X}_i,$$
$$int(\mathcal{X}_i) \cap int(\mathcal{X}_j) = \emptyset, i \ne j, \forall i, j \in \mathcal{I}_N.$$

Such a polyhedral partition will be denoted as $\mathcal{P}_N(\mathcal{X})$.

Definition 2. *Consider a polyhedron $\mathcal{X} \subset \mathbb{R}^{n_x}$ with a related polyhedral partition $\mathcal{P}_N(\mathcal{X})$, a piecewise affine (PWA) function over $\mathcal{P}_N(\mathcal{X})$ is defined as:*

$$f_i : \mathbb{R}^{n_x} \to \mathbb{R} \tag{2}$$
$$f_i(x) = a_i^T x + b_i, \forall x \in \mathcal{X}_i. \tag{3}$$

Following the classic EMPC design Bemporad et al. (2002), a PWA control law is obtained based on the solution of a multi-parametric quadratic programming (mp-QP) problem with the following structure:

$$J(x(k)) := \min_u \quad 1/2 u^T H u + x^T(k) F u \tag{4a}$$
$$s.t. \quad Gu \le W + Sx(k). \tag{4b}$$

The solution $u = g_i^T x(k) + f_i$ for all $x(k) \in \mathcal{X}_i, i \in \mathcal{I}_N$ is a PWA funciton defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the state set $\mathcal{X}$. The control input $u(k)$ is the first component of the input sequence u.

The polyhedral partition $\mathcal{P}_N(\mathcal{X})$ and related control law $u(x)$ have to be stored, and the online computational effort is concentrated on the evaluation of the appropriate local affine function. Such a look-up table implementation of a nonlinear control has apparent advantages in safety certification in industrial application. However, their use in dedicated hardware devices comes inevitably accompanied by quantization operations, which are used on the representation of the state regions and the associated PWA control laws (Knyazev et al. (2015)).

In Koduri et al. (2016), Koduri et al. (2017a), authors studied the effectiveness of quantized operations on the PWA control laws and proposed a sensitivity margin to reduce these unflattering effects. After a short analysis of the current work in the following subsection, a question worth exploring is raised.

2.2 Existing results on PWA sensitivity

In the following, let us consider a PWA control law and associated state regions designed for the stabilization of the system (1). The vertex representation of the state region $\mathcal{X}_i$ is given as:

$$\mathcal{X}_i = conv\{v_{i,1}, v_{i,2}, \dots, v_{i,r_i}\}, i \in \mathcal{I}_N, \tag{5}$$

where $\mathcal{I}_N$ and r_i are the region number set and the number of vertices of $\mathcal{X}_i$, respectively.

As discussed in Knyazev et al. (2015), the quantization is inevitable in the process of numerical representation of the regions in the state space and consequently on the PWA control law evaluation. On one hand, the quantization operations can reduce the storage requirements and consequently the evaluation. On the other hand, quantization-induced changes in control laws need to be carefully analyzed as they may lead to non-unicity at the evaluation stage, loss of invariance properties, and even system instability. In this respect, in Koduri et al. (2017a), authors investigated the admissible perturbation set for vertices in the vertex representation of state regions. In particular, a sensitivity margin (SM) was introduced to describe the vertex sensitivity after perturbation in terms of a set that can guarantee the non-overlapping property of the resulting partition.

Definition 3. *Consider the polyhedral partitions $\mathcal{P}_N(\mathcal{X})$ of a PWA control law for system (1) with each region given by its vertex representation (5). Let $v \in \mathbb{R}^{n_x}$ be a vertex of the regions composing $\mathcal{P}_N(\mathcal{X})$ and denote $\Theta(v)$ as the set of indices of all polyhedral regions having v as a vertex:*

$$\Theta(v) = \{j \in \mathcal{I}_N | v \in \mathcal{V}(\mathcal{X}_j)\}.$$

The ***sensitivity margin*** *of a vertex v within $\mathcal{P}_N(\mathcal{X})$ is defined as the set $\Phi \subset \mathbb{R}^{n_x}$ containing all the points $(v+\delta v)$ for which the collection of sets*

$$\hat{\mathcal{X}}_j = conv\{\mathcal{V}(\mathcal{X}_j) \setminus \{v\}, v + \delta v\}, \forall j \in \Theta(v),$$
$$\hat{\mathcal{X}}_j = \mathcal{X}_j, \forall j \in \mathcal{I}_N \setminus \Theta(v),$$

represents a polyhedral partition: $\hat{\mathcal{P}}_N(\mathcal{X}) = \{\hat{\mathcal{X}}_1, \dots, \hat{\mathcal{X}}_N\}$.

Each vertex of the regions within the polyhedral partition $\mathcal{P}_N(\mathcal{X})$ has an SM. In the following, the SM of a given vertex v will be denoted Φ for simplicity. If the SMs of different vertices v^i and v^j are dealt with, they will be denoted Φ^i and Φ^j to avoid ambiguity.

Clearly, the SM contains the original vertex, i.e., $v \in \Phi$, and its uniqueness and structural properties can be fully characterized as resumed by the following result.

Lemma 1. *(Koduri et al. (2017b)) Consider the polyhedral partition $\mathcal{P}_N(\mathcal{X}) \subset \mathbb{R}^{n_x}$ composed by the regions $\mathcal{X}_i, i \in \mathcal{I}_N$. For $v \in \mathcal{V}(\mathcal{X}_i)$, the SM is unique and the corresponding set Φ is a polyhedron.*

If the vertex v is perturbed within its SM region, the resulting polyhedral partition will preserve the non-overlapping property. However, SM merely defines the sensitivity of a single vertex, whereas the study of coupled vertex sensitivity of a pair of vertices is an open problem and represents a challenge from the computational point of view.

It should be mentioned that Koduri's work explored and eventually implemented perturbation studies of multiple vertices, but the essence of this sensitivity analysis was built on the sequential perturbation of the vertices. This essentially iterative procedure increases the conservativeness of the robustness margin and is highly dependent on the order (priority) of processing the multiple vertices.

In order to address all these issues and reduce the conservativeness of joint vertex sensitivity, a method is proposed next to define and characterize the sensitivity margin of two vertices concomitantly.

3. MAIN RESULT

3.1 Coupled vertex sensitivity definitions

This subsection proposes a vertex sensitivity analysis for the simultaneous perturbation of two vertices starting from the former SM and introduces the coupled vertices sensitivity (CVS), aiming at reducing the conservativeness of multiple vertices perturbation analysis in calculating vertex sensitivity. It expands, then, the model of vertex perturbation and provides ideas for further study of half-space perturbation.

Definition 4. *Consider the polyhedral partition $\mathcal{P}_N(\mathcal{X}) \in \mathbb{R}^{n_x}$ with each region given by its vertex representation $\mathcal{X}_i = conv\{v_{i,1}, \dots, v_{i,r_i}\}, i \in \mathcal{I}_N$. Let $v^1, v^2 \in \mathbb{R}^{n_x}$ be vertices within $\mathcal{P}_N(\mathcal{X})$ and denote the set of indexes of polyhedral regions having v^1 or v^2 as a vertex as follows:*

$$\Theta^{12}(v^1, v^2) = \{j \in \mathcal{I}_N | v^1 \in \mathcal{V}(\mathcal{X}_j), v^2 \in \mathcal{V}(\mathcal{X}_j)\}, \quad (6)$$

$$\Theta^{1}(v^1, v^2) = \{j \in \mathcal{I}_N | v^1 \in \mathcal{V}(\mathcal{X}_j), v^2 \notin \mathcal{V}(\mathcal{X}_j)\}, \quad (7)$$

$$\Theta^{2}(v^1, v^2) = \{j \in \mathcal{I}_N | v^1 \notin \mathcal{V}(\mathcal{X}_j), v^2 \in \mathcal{V}(\mathcal{X}_j)\}. \quad (8)$$

The pair of sets (Ψ^1, Ψ^2) is describing a ***coupled vertex sensitivity*** *of vertices pair (v^1, v^2) within the polyhedral partition $\mathcal{P}_N(\mathcal{X})$ if the following conditions are satisfied:*

- $\Psi^1 \subset \mathcal{X} \subset \mathbb{R}^{n_x}, \Psi^2 \subset \mathcal{X} \subset \mathbb{R}^{n_x}$,
- *$v^1 \in \Psi^1$ and $v^2 \in \Psi^2$,*
- *for all $(v^1 + \delta v^1) \in \Psi^1$ and $(v^2 + \delta v^2) \in \Psi^2$, the collection of sets $\hat{\mathcal{P}}_N(\mathcal{X}) = \{\hat{\mathcal{X}}_i\}_{\forall i \in \mathcal{I}_N}$.*

$$\hat{\mathcal{X}}_j = conv\{\mathcal{V}(\mathcal{X}_j) \setminus \{v^1, v^2\}, v^1 + \delta v^1, v^2 + \delta v^2\}, \forall j \in \Theta^{12},$$

$$\hat{\mathcal{X}}_j = conv\{\mathcal{V}(\mathcal{X}_j) \setminus \{v^1\}, v^1 + \delta v^1\}, \forall j \in \Theta^{1},$$

$$\hat{\mathcal{X}}_j = conv\{\mathcal{V}(\mathcal{X}_j) \setminus \{v^2\}, v^2 + \delta v^2\}, \forall j \in \Theta^{2},$$

$$\hat{\mathcal{X}}_j = \mathcal{X}_j, \forall j \in \mathcal{I}_N \setminus \Theta^{12} \cup \Theta^1 \cup \Theta^2$$

represents a polyhedral partition, where Θ^* denotes the set $\Theta^*(v^1, v^2)$.

For this new sensitivity notion, one can establish a series of properties underlying the non-unicity of the CVS and the relationships with the SM of each vertex in the pair considered independently.

Proposition 1. *Consider two vertices v^1 and v^2 within a polyhedral partition $\mathcal{P}_N(\mathcal{X})$. The following properties hold:*

- *If $\Theta^{12}(v^1, v^2)$ is an empty set, the perturbation of v^2 has no effect to the SM of v^1.*
- *Given the SM Φ^1 of the vertex v^1, the pair $(\Phi^1, \{v^2\})$ represents a CVS for the pair of vertices (v^1, v^2).*
- *The CVS of the pair of vertices (v^1, v^2) within the polyhedral partition $\mathcal{P}_N(\mathcal{X})$, given by the pair of sets (Ψ^1, Ψ^2) may not be unique. Two trivial examples are $(\Phi^1, \{v^2\})$ and $(\{v^1\}, \Phi^2)$.*
- *If (Ψ_1^1, Ψ_1^2) and (Ψ_2^1, Ψ_2^2) are two CVS of (v^1, v^2) then the pair of sets*
$$(\Psi_1^1 \cup \Psi_2^1, \Psi_1^2 \cap \Psi_2^2) \text{ or } (\Psi_1^1 \cap \Psi_2^1, \Psi_1^2 \cup \Psi_2^2)$$
are a CVS of the pair of vertices (v^1, v^2). As a particular case, the pair $(\Psi_1^1 \cap \Psi_2^1, \Psi_1^2 \cap \Psi_2^2)$ is a CVS of (v^1, v^2) too.

These properties are provided here without formal proof, but the arguments supporting the claims construct the CVS as described in the following subsection.

3.2 Constructive approach for the CVS

Taking into account that the CVS of a pair of vertices is elaborated starting from the essential SM of the vertices taken independently, we analyzed the interplay of their variation in this subsection. From a theoretical point of view, a proposition will formally present the operations related to the CVS construction. Finally, an algorithm is provided to calculate the CVS.

Consider v^1 and v^2, two vertices of the polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the PWA system (1). The aim is to establish formal conditions to prevent the phenomenon of overlapping of the regions composing the PWA partition. This non-overlapping property has been considered in the computation process of Φ^1, the SM for v^1 considered independently (i.e., with fixed positions of the remaining vertices). The objective here is to point out the role of the uncertainty on a different vertex v^2 acting in this robustness analysis.

Let us begin by considering all the vertices of $\mathcal{P}_N(\mathcal{X})$ excepting v^1 as fixed and denote their position by $\bar{v}^i, \forall i \neq 1$. In Koduri et al. (2017a), the linear affine inequalities defining the SM have been characterized thus leading to a polyhedral set $\Phi^1 = \{x | H^1 x \leq b^1\}$.

$$\Phi^1 = \underbrace{\{x \in \mathbb{R}^n | H^{1,1} x \leq b^{1,1}\}}_{\Gamma^1} \cap \underbrace{\{x \in \mathbb{R}^n | H^{1,2} x \leq b^{1,2}\}}_{\Delta^1}$$

where the reordering and the separation of halfspace has been operated such that:

$$H^1 = \begin{bmatrix} H^{1,1} \\ H^{1,2} \end{bmatrix} \text{ and } b^1 = \begin{bmatrix} b^{1,1} \\ b^{1,2} \end{bmatrix}$$

with $H^{1,1}\bar{v}^2 = b^{1,1}$ and $H^{1,2}\bar{v}^2 < b^{1,2}$, where $v^2 \neq v^1$ is a vertex within $\mathcal{P}_N(\mathcal{X})$ subject to a joint disturbance around its nominal position $\bar{v}^2$, and the related $\Theta^{12}(v^1, v^2)$ is not empty. By selecting halfspace saturated by $\bar{v}^2$, one can construct the polyhedral set

$$\Gamma^1 = \{x | H^{1,1} x \leq b^{1,1}\},$$

which is representative for the effect of variation of v^2 on the robustness margin for v^1. Obviously, $\Phi^1 \subset \Gamma^1$ and

Figure 1 provides a graphical illustration of the sets defined in the present case. It should be specified that, due to $\Phi^1 \subset \mathcal{X}$, for the convenience of graphing Γ^1, in Figure 1, we only showed the intersection of Γ^1 and $\mathcal{X}$ to denote Γ^1.

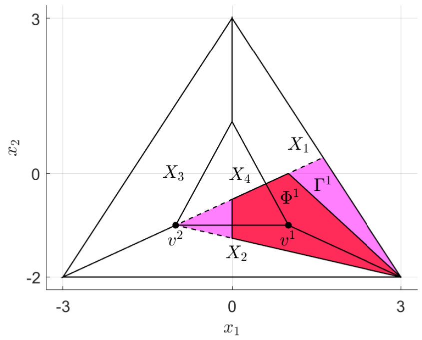


Fig. 1. A polyhedral partition with four regions, a vertex v^1 and the SM Φ^1 of v^1. Polyhedral Γ^1 depicted by magenta dotted zone denotes the interaction of v^2 with v^1.

Let us consider now a perturbation affecting the representation of the vertex v^2 to $\hat{v}^2 = \bar{v}^2 + \delta v^2$. In order to account for this perturbation in the definition of the SM for v^1, one has to consider the set Φ^1 in terms of its parameterization with respect to the position of the vertex $\hat{v}^2$. Thus in the following, we will denote by $\Phi^1(\hat{v}^2)$ the SM of v^1 assumed that the vertex v^2 is displaced from the position $\bar{v}^2$ to $\hat{v}^2$. Following the same convenience of notation, after recomputing the set $\Phi^1(\hat{v}^2)$, one can subsequently obtain $\Gamma^1(\hat{v}_2)$. Figure 2 illustrated the position of these regions.

Our objective is to obtain a joint sensitivity for a pair of vertices (v^1, v^2) and the first operation is to extend the CVS from $(\Phi^1(\bar{v}^2), \{\bar{v}^2\})$ towards the CVS $(\Phi^1(\{\bar{v}^2, \hat{v}^2\}), \{\bar{v}^2, \hat{v}^2\})$.

The next step is to clarify the relationship between $\Phi^1(\bar{v}^2)$ and $\Phi^1(\hat{v}^2)$ on one side and $\Phi^1(\{\bar{v}^2, \hat{v}^2\})$ on the other side. It is easy to observe from the definition of SM sets that:

$$\Phi^1(\{\bar{v}^2, \hat{v}^2\}) = \Phi^1(\bar{v}^2) \cap \Phi^1(\hat{v}^2). \tag{9}$$

On the left-hand side, $\Phi^1 = \Phi^1(\{\bar{v}^2, \hat{v}^2\})$ represents the SM of v^1 and should be understood as the margin of its variation independent of the position of v^2 at $\bar{v}^2$ or $\hat{v}^2$. On the right-hand side, the effect of an undesirable perturbation of v^2 the characterization of the SM of v^1 is expressed by the intersection of the SM as long as the admissible perturbations need to be part of both sets.

The main contribution with respect to these developments is the extension of the v^2 displacement within the set $conv\{\bar{v}^2, \hat{v}^2\}$. Based on the construction principles for the sets $\Gamma^1(.)$ which represents the collection of active constraints for the possible variation of v^2 one can note that whenever $\Gamma^1(v^2)$ satisfies

$$\Gamma^1(\bar{v}^2) \cap \Gamma^1(\hat{v}^2) \subset \Gamma^1(v^2), \forall v^2 \in conv\{\bar{v}^2, \hat{v}^2\},$$

then Φ^1, the SM of v^1, will not be affected by the perturbation of v^2 within the set $conv\{\bar{v}^2, \hat{v}^2\}$. Consequently, $(\Phi^1(\{\bar{v}^2, \hat{v}^2\}), conv\{\bar{v}^2, \hat{v}^2\})$ is a CVS of (v^1, v^2) as illustrated in Figure 3.

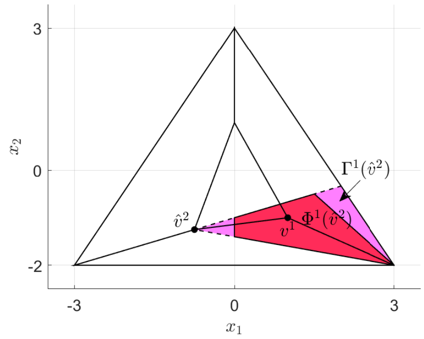


Fig. 2. With the perturbation of $v^2 \rightarrow \hat{v}^2$, the SM of v^1 changed as: $\Phi^1 \rightarrow \Phi^1(\hat{v}^2)$ (red zone), the related effect set changed as: $\Gamma^1 \rightarrow \Gamma^1(\hat{v}^2)$ (magenta zone)
.

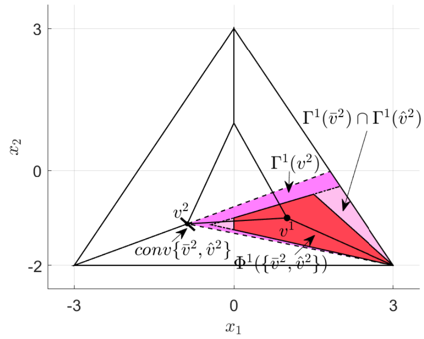


Fig. 3. The effect of v^2 to the SM of v^1: $\Phi^1(\{\bar{v}^2, \hat{v}^2\}) \subset \Gamma^1(\bar{v}^2) \cap \Gamma^1(\hat{v}^2) \subset \Gamma^1(v^2), \forall v^2 \in conv\{\bar{v}^2, \hat{v}^2\}$.

Proposition 2. *Consider the polyhedral partitions $\mathcal{P}_N(\mathcal{X})$ of a PWA control law for system (1) and suppose all the vertices are at their nominal positions excepting the pair (v^1, v_i^2). $\Phi^1(v_i^2)$ is the SM of v^1 and $\Gamma^1(v_i^2)$ is the parameterized set reflecting the effect of v_i^2 to v^1. The pair of sets $\left(\Phi^1(\{v_1^2, v_2^2, \ldots, v_m^2\}), conv\{v_1^2, v_2^2, \ldots, v_m^2\}\right)$ is a CVS for (v^1, v_i^2) if:*

$$\Gamma^1(\tilde{v}^2) \supseteq \bigcap_{l=1}^{m} \Gamma^1(v_l^2), \forall \tilde{v}^2 \in conv\{v_1^2, v_2^2, \ldots, v_m^2\} \tag{10}$$

Proof. $\left(\Phi^1(\{v_1^2, v_2^2, \ldots, v_m^2\}), conv\{v_1^2, v_2^2, \ldots, v_m^2\}\right)$ represents a CVS for (v^1, v^2) if

$$\Phi^1(\{v_1^2, v_2^2, \ldots, v_m^2\}) = \bigcap_{\tilde{v}^2 \in conv\{v_1^2, v_2^2, \ldots, v_m^2\}} \Phi^1(\tilde{v}^2). \tag{11}$$

By definition

$$\Phi^1(\{v_1^2, v_2^2, \ldots, v_m^2\}) \equiv \bigcap_{l=1}^{m} \Phi^1(\tilde{v}_l^2)$$

and

$$\bigcap_{l=1}^{m} \Phi^1(\tilde{v}_l^2) \supseteq \bigcap_{\tilde{v}^2 \in conv\{v_1^2, v_2^2, \ldots, v_m^2\}} \Phi^1(\tilde{v}^2).$$

Thus, in order to prove (11) one needs to prove that

$$\Phi^1(\tilde{v}^2) \supseteq \bigcap_{l=1}^{m} \Phi^1(v_l^2)$$

forall $\tilde{v}^2$ in $conv\{v_1^2, v_2^2, \ldots, v_m^2\}$. This can be rewritten as

$$\Gamma^1(\tilde{v}^2) \cap \Delta \supseteq \bigcap_{l=1}^{m} \Gamma^1(v_l^2) \cap \Delta$$

Clearly, the condition (10) represents a sufficient condition and the Proposition is proved. □

Figure 3 depicts the central notions of the proof and points to the essence of the conditions embedded in the Proposition 2 in the particular case of $m = 2$ extreme points for the variation of v^2.

Remark 1. *The condition* (10) *provides guarantees for the existence of a joint sensitivity margin, but the essential feature to be underlined is that neighbour vertices influence each other, and the joint sensitivity can be analyzed using the sensitivity margin at the extreme points of disturbance for the neighbour vertex. Furthermore, the joint sensitivity margin opens the way for the treatment of the sensitivity margin for the H-representation of the polyhedral partition of the PWA control laws with obvious practical advantages in the implementation.*

Algorithm 1 presented next proposes a constructive procedure to calculate the CVS for a pair of vertices of a polyhedral partition (v^1, v^2). It alternates the construction of SM for independent vertices and integrates the results to obtain a non-degenerate pair of sets representing a CVS. It should be remarked that the construction of CVS is not unique as it was the case for SM of single vertices. Consequently, the output depends on the initialization, in this case, the processing order for the chosen vertices.

Algorithm 1 Calculating the CVS (Ψ^1, Ψ^2)

Require: $\mathcal{X} = \cup_{i=1}^{N} \mathcal{X}_i, i \in \mathcal{I}_N$ and $v^1, v^2 \in \mathcal{P}_N(\mathcal{X})$.
Ensure: (Ψ^1, Ψ^2) as CVS for the pair (v^1, v^2).
1: Fix $\bar{v}^1$ in the PWA explicit formulation and calculate the $\Phi^2(\bar{v}^1)$, the SM for v^2.
2: Initialize Ψ^2 to a subset of Φ^2 which satisfy $v^1 \notin \Psi^2$.
3: Get the V-representation of set Ψ^2, which it can be rewrote as $\Psi^2 = conv\{v_1^2, \ldots, v_{r_2}^2\}$.
4: **for** $i = 1, \ldots, r_2$ **do**
5: Fix $v^2 \to v_i^2$ and calculate the SM $\Phi^1(v_i^2)$ for v^1.
6: **end for**
7: $\Psi^1 = \cap_{i=1}^{r_2} \Phi_i^1$.
8: **return** The CVS of (v^1, v^2) as (Ψ^1, Ψ^2).

4. ILLUSTRATIVE EXAMPLES

Consider a discrete-time linear system mentioned at Koduri et al. (2017a):

$$x_{k+1} = Ax_k + Bu_k \tag{12}$$

with the states and inputs constraints

$$-5 \leq [1\ 0]x \leq 5, \|u\|_\infty \leq 5.$$

The system parameter matrices are:

$$A = \begin{bmatrix} 1.4 & 0 \\ 1.8 & -1.1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}$$

The PWA controller has been obtained in an explicit form (MPT 3.0 toolbox Herceg et al. (2013)) for this numerical example with parameters parameters $Q = \mathbb{I}_2$, $R = 1$, a prediction step equals 2, an LQR terminal set and a terminal state cost

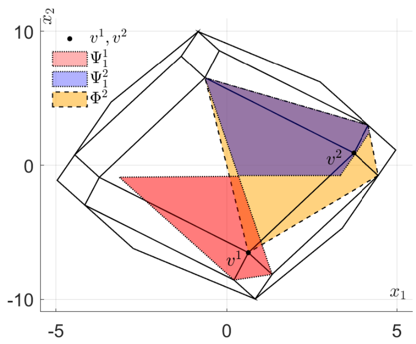
$$P = \begin{bmatrix} 5.6902 & 0.4862 \\ 0.4862 & 3.1099 \end{bmatrix}.$$

Two scenarios are discussed next to illustrate the characteristics of the coupled vertex sensitivity.

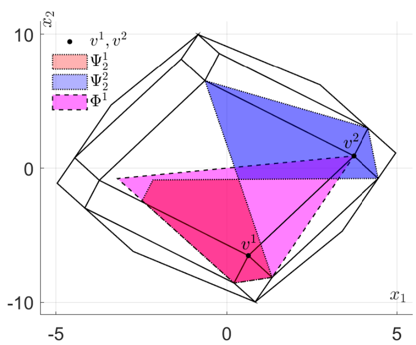
Scenario 1**:**

In a first approach, the sensitivity analysis is based on constructing the CVS for a pair of vertices belonging to the related polyhedral partition to illustrate the efficiencies of the concept and the realization based on Algorithm 1.

Firstly, let us recall the result of the nominal control design, which leads to 13 linear-affine controllers and their associated state space partitions. Next, let us select a pair vertices (v^1, v^2) found at the nominal positions: $\bar{v}^1 = [0.6361 \quad -6.5235]^T$ and $\bar{v}^2 = [3.7291 \quad 0.9076]^T$. Based on Algorithm 1, CVS (Ψ^1, Ψ^2) was calculated and depicted in Figure 4. The graphical representation underlines the advantage of the CVS representation in comparison with the original SM of the vertices taken independently.



(a) CVS (Ψ_1^1, Ψ_1^2) for (v^1, v^2) with the order of processing starting at v^1 and followed up by v^2



(b) CVS (Ψ_2^1, Ψ_2^2) for (v^1, v^2) with the order of processing starting at v^2 and followed up by v^1

Fig. 4. Different CVSs obtained for the pair of vertices (v^1, v^2).

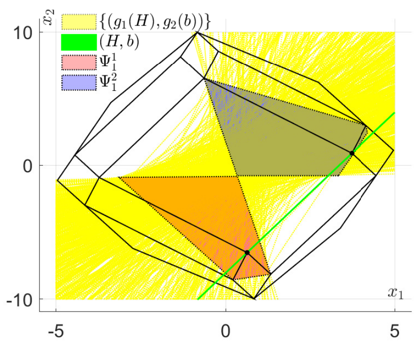
As expected, the output of the Algorithm 1 is dependent on the processing order and the Figure 4.a shows the result of calculating Ψ^1 first, while Figure 4.b places Ψ^2 in the first processing stage. The non-uniqueness of the joint sensitivity margin represents a rich family for the treatment of the quantization error in the PWA implementation.

Scenario 2:

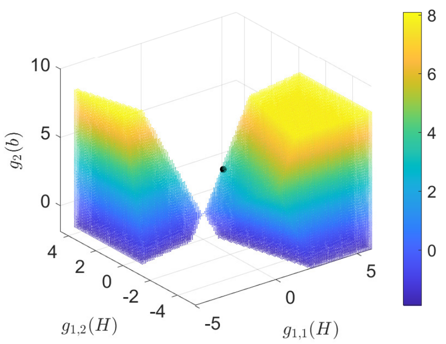
Using the analysis performed in Scenario 1 and given the fact that the pair of vertices are interdependent in the sensitivity analysis, it opens the way to the characterize the admissible perturbations in the representation of the halfspace, which saturates the two vertices.

Similar with the quantization of vertices, one can consider quantization functions $g_1(H) = H + \Delta H$ and $g_2(b) = b + \Delta b$, which were used to describe the perturbation in the halfspace representation, with $\|\Delta H\|_\infty \leq 0.1$, $\|\Delta b\|_\infty \leq 0.1$, $Hv^1 = b$, and $Hv^2 = b$.

We considered the computed CVS as constraints for the admissible halfspace perturbation. In practice, if the intersection of the half-plane $(g_1(H), g_2(b)) : \{x | g_1(H)x = g_2(b)\}$ has a non-empty intersection with both sets composing the CVS then the perturbation $(g_1(H), g_2(b))$ is admissible for the half-plane (H, b). Obviously, the original half-plane is admissible by the fact that $\bar{v}^1 \in \Psi^1$ and $\bar{v}^2 \in \Psi^2$. Figure 5, illustrates graphically the admissible half-planes, first in the original partition space and secondly in the space of parameters of the halfspace.



(a) The admissible perturbation position set $\{(g_1(H), g_2(b))\}$ for hyperplane (H, b) who contains v^1 and v^2, denoted by yellow lines.



(b) Admissible variation of the coefficients characterizing the half-plane under study. $g_{1,1}(H)$ denotes the value of the first element of $g_1(H)$ and black dot represents the value of H and b.

Fig. 5. Relationship between the CVS (Ψ_1^1, Ψ_1^2) and the admissible perturbation $(g_1(H), g_2(b))$ of the half-plane saturated by these two vertices.

5. CONCLUSION

The paper analyzes the PWA control laws regarding the sensitivity of their representation. The geometrical properties of the partition inherited from the control design can suffer from errors in the numerical representation of the coefficients. As the main contribution, it has been shown that the vertex representation of the partition can be characterized by a coupled sensitivity measure. The CVS goes beyond state of the art, which handles the sensitivity for each vertex taken independently.

Several avenues for further development are open. We can mention the non-unicity of the pairs of sets describing the coupled sensitivity margins, which deserves further attention to define an appropriate criterion for the selection or the interest in the constructive approaches for their construction. On a broader scope, the gap between the sensitivity margins for the vertex perturbation and the halfspace perturbation remains widely uncovered.

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