



Technische  
Universität  
Braunschweig

Institut für  DS  
Dynamik und Schwingungen



## Learning of Frequency Response Functions in Electrical Engineering

with N. Georg<sup>1,2</sup>, J. Bect<sup>3</sup> and S. Schöps<sup>2</sup>; SCEE 2022

Ulrich Römer<sup>1</sup>, July 12, 2022

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# Parametric Problem

- We address linear parametric problems in the form

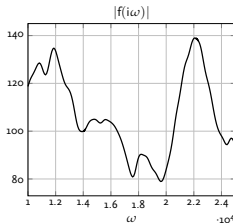
$$(\mathbf{K} + s\mathbf{D} + s^2\mathbf{M})\mathbf{u} = \mathbf{g}$$

- Stiffness, damping and mass matrix  $\mathbf{K}, \mathbf{D}, \mathbf{M}, \in \mathbb{R}^{n_h \times n_h}$
- Complex frequency variable  $s = i\omega$ , where  $\omega \in \Omega \subset \mathbb{R}^+$

- Quantity of Interest (QoI)

$$f: \mathbb{C} \rightarrow \mathbb{C}, s \mapsto \mathcal{J}(\mathbf{u}(s))$$

- Training data set  $\{\omega_i, f(i\omega_i)\}_{i=1}^n$

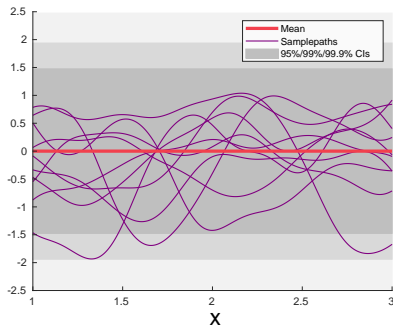


- **Goal:** Approximation of FRFs based on a few training points
- **Approach:** Complex Gaussian Process (GP) / Kernel methods

# Gaussian Process Regression and Kernel Methods

$$f(x) \sim \text{GP}(m(x), k(x, x'))$$

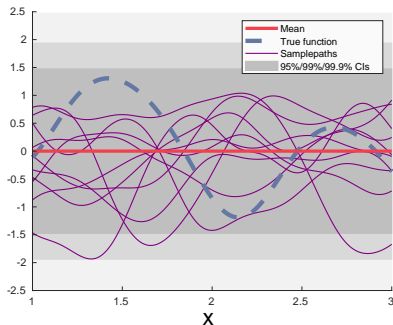
- Mean function  $m(x)$
  - Covariance function  $k(x, x')$
- Gaussian Process Regression (GPR): infer unobserved values by conditioning on training data
- Approximation of complex functions
    1. Use separate GP approximations for real and imaginary part
    2. Use complex GP with
      - Covariance function  $k(x, x') = \mathbb{E}[(f(x) - m(x))(f(x) - m(x'))^*)]$
      - Pseudo-covariance function  $c(x, x') = \mathbb{E}[(f(x) - m(x))(f(x) - m(x')))]$
  - Posterior mean: minimum norm interpolant in the RKHS with kernel  $k$



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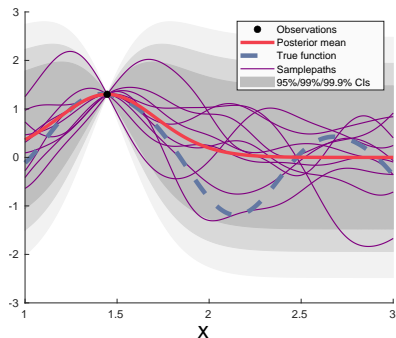
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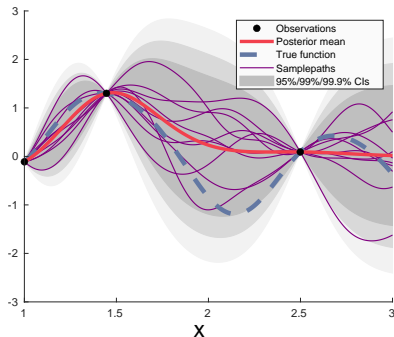
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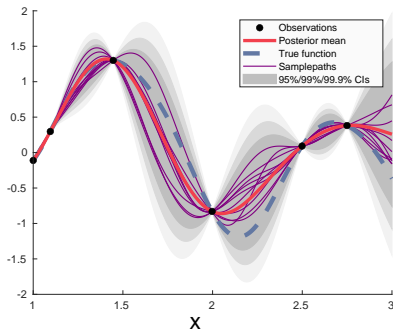
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# Contents

- Introduction
- **Complex/Real Kernel Methods**
- Numerical Results



# A Framework for Complex Kernel / GP Approximation

- FRFs are complex-valued functions  $f: \mathcal{S} \rightarrow \mathbb{C}, s \mapsto f(s)$
- Symmetry condition  $f(s^*) = f(s)^*, \forall s \in \mathcal{S}$  requires real vector spaces

## Definition: complex/real RKHS

A real Hilbert space of complex valued functions (denoted  $H$ ) over  $\mathcal{S}$ , with continuous evaluation functionals  $\delta_s(f) = f(s)$ .

- A complex/real RKHS is characterized by
  - a kernel  $k$  and pseudo-kernel  $c$
- **Complex GPs** R. Boloix-Tortosa et al., "Complex Gaussian Processes for Regression", IEEE Trans Neural Netw Learn Syst, 2018
- **Widely-linear estimation** B. Picinbono and P. Chevalier, "Widely linear estimation with complex data", IEEE Trans Signal Process, 1995

# A Framework for Complex Kernel / GP Approximation

## Kernel interpolation in a complex/real RKHS

Let  $s_1, \dots, s_n \in \mathcal{S}$  and  $y_1, \dots, y_n \in \mathbb{C}$ . There exists a unique interpolant  $f \in H$  such that  $f(s_i) = y_i, i = 1, \dots, n$ , if and only if

$$K_n \gamma + C_n \gamma^* = y, \quad \text{with } K_n = (k(s_i, s_j))_{i,j}, C_n = (c(s_i, s_j))_{i,j}$$

admits a solution  $\gamma \in \mathbb{C}^n$ .

The function  $f_n = \sum_{i=1}^n \gamma_i k(\cdot, s_i) + \gamma_i^* c(\cdot, s_i)$ :

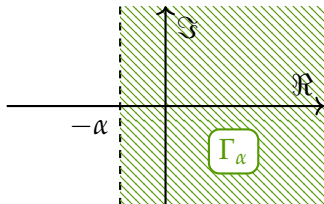
- is the unique interpolant with minimal norm in  $H$
- is the conditional mean prediction of the GP with covariance  $k$  and pseudo-covariance  $c$

# Complex/Real RKHS with Symmetry Condition

- If all  $f \in H$  satisfy  $f(s^*) = f(s)^*$ , then
  - kernel and pseudo-kernel are related as  $c(s, s_0) = k(s, s_0^*)$
  - interpolation requires  $y_j = y_i^*$  for all  $s_j = s_i^*$
- Example: Hardy space  $H^2(\Gamma_\alpha)$

Assumption:  $f \in H^2(\Gamma_\alpha)$ , where

$$\Gamma_\alpha := \{s \in \mathbb{C} \mid \Re[s] > -\alpha\}, \alpha > 0$$



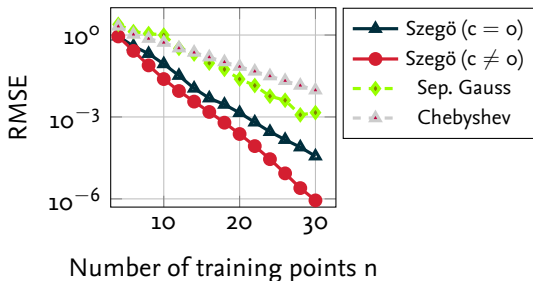
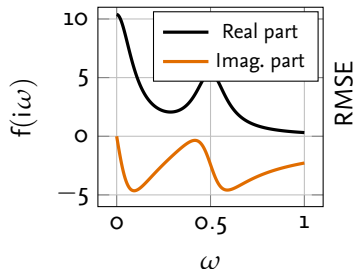
$H^2(\Gamma_\alpha)$  is an RKHS with the kernel  $k$  (Szegő) and pseudo-kernel  $c$

$$k(s, s_0) = \frac{1}{2\pi(2\alpha + s + s_0^*)} \quad c(s, s_0) = \frac{1}{2\pi(2\alpha + s + s_0)}$$

See also J. Lataire and T. Chen. "Transfer function and transient estimation by Gaussian process regression in the frequency domain", Automatica, 2016

# Choice and Effect of Pseudo-Covariance

- Comparison of different interpolation techniques:
  - Complex/real interpolation
  - Separate interpolation of real and imaginary part
- Numerical example: third order rational function  $f \in H^2(\Gamma_\alpha)$ ,  $\omega \in [0, 1]$



# Combining Kernel and Rational Approximation

- Problem: slow convergence of complex/real kernel interpolation  $f_n(i\omega; \alpha)$  if function has a few dominant poles
- Idea: combine kernel approach with rational basis functions<sup>1</sup>

$$f_n^{(K)}(i\omega; \mathbf{p}, \alpha) = f_n(i\omega; \alpha) + \sum_{i=1}^K \frac{1}{i\omega - p_i} r_i + \frac{1}{i\omega - p_i^*} r_i^*$$

- Approach: heuristic algorithm generating models  $\{f_n^{(K)}\}_{K=0}^{K_{\max}}$  by tuning hyper-parameter  $\alpha$  and pole pairs
- Model selection using a novel criterion based on leave-on-out cross-validation with an additional instability penalty

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<sup>1</sup>B. Gustavsen and A. Semlyen, "Rational approximation of frequency domain responses by vector fitting", IEEE Trans. Power Delivery (1999)

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# Electric Circuit

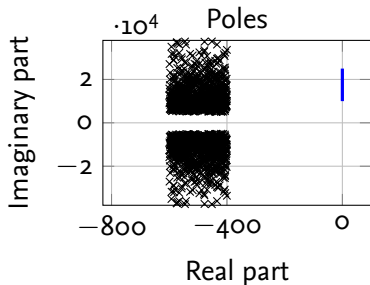
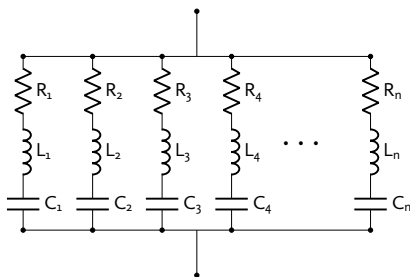
- Parallel connection of  $n$  underdamped series RLC circuits

$$Y(i\omega) = \sum_{i=1}^n \frac{i\omega}{(i\omega)^2 L_i + i\omega R_i + C_i^{-1}}, \quad \omega \in [10 \text{ kHz}, 25 \text{ kHz}]$$

- Setting 1:  $n_1 = 1000$  random series RLC elements

$$C_i \sim \mathcal{U}(1, 20) \mu\text{F}, \quad L_i \sim \mathcal{U}(0.1, 2) \text{ mH}, \quad R_i = L_i(1 + \mathcal{U}(-0.2, 0.2)) \frac{\Omega}{\text{mH}}$$

- Setting 2: two additional circuit elements with very low damping



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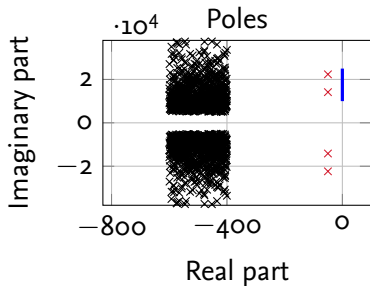
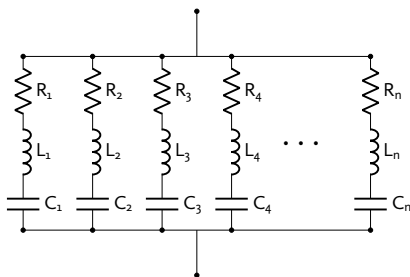
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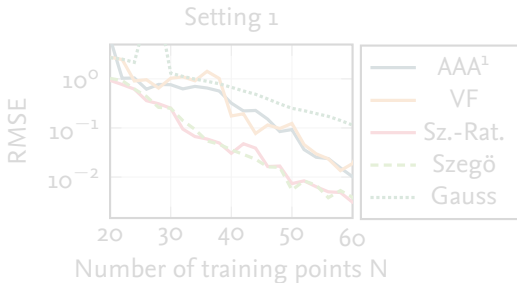
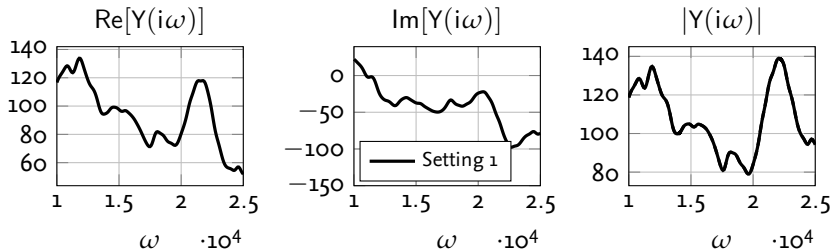
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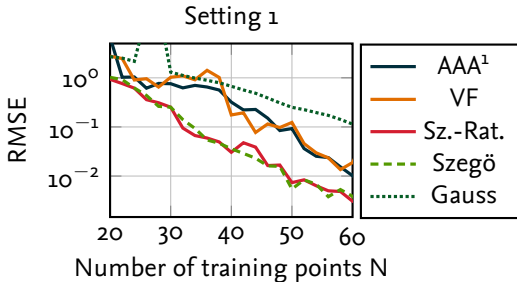
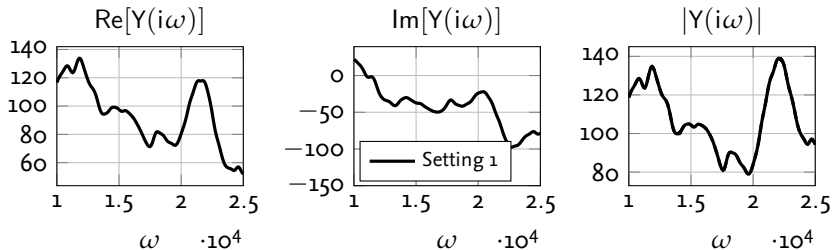


# Electric Circuit - Setting 1



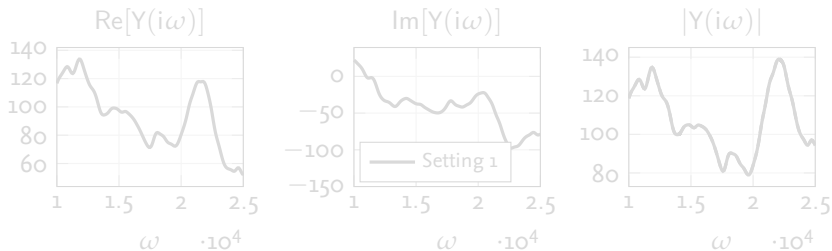
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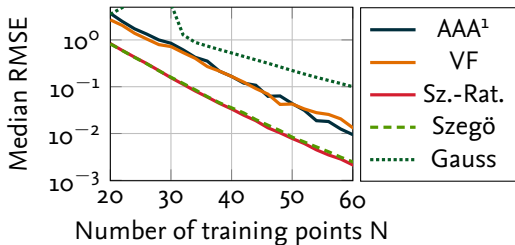


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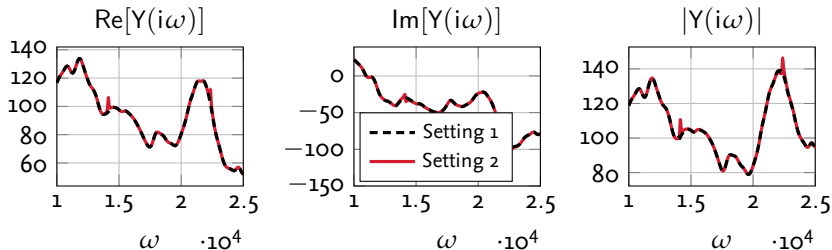


Setting 1 (100 runs)

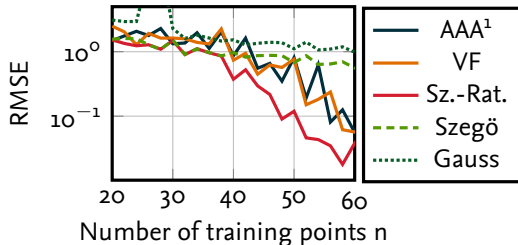


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# Electric Circuit - Setting 2

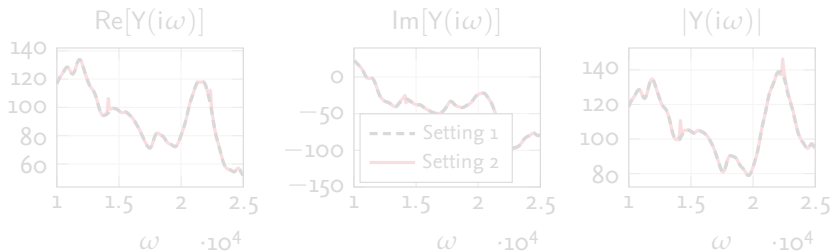


Setting 2 (1 run)

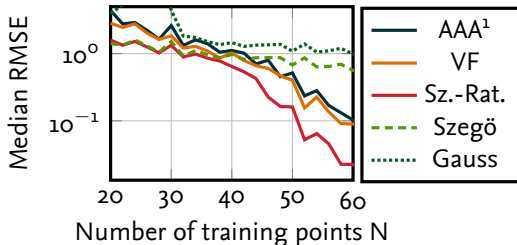


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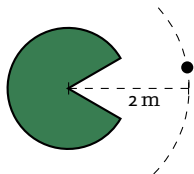


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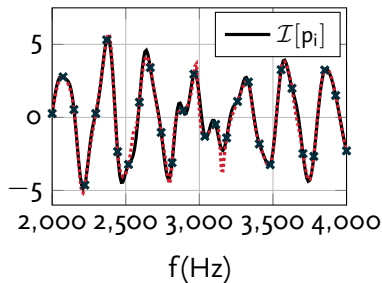
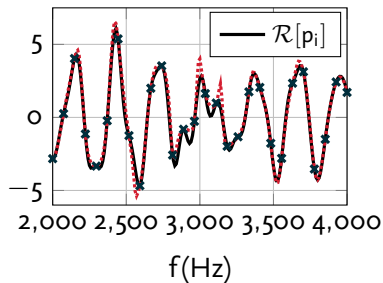


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# Benchmark Problem: Acoustic Scattering

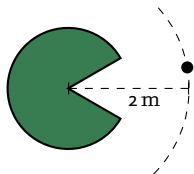


- PACMAN shape: vibrating surface
- Acoustic pressure  $p_i$  at black dot
- Approx. with  $n$  training points (blue crosses)
- **Method: AAA (rational approximation)**

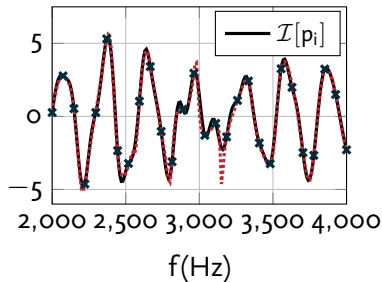
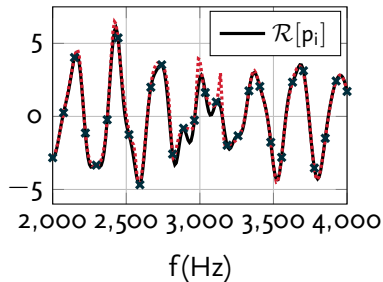


H. Ziegelwanger, P. Reiter, "The PAC-MAN model: Benchmark case for linear acoustics in computational physics," J Comput Phys, 2017

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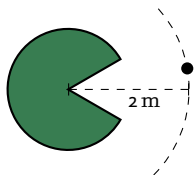


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- **Vector Fitting (rational approximation)**

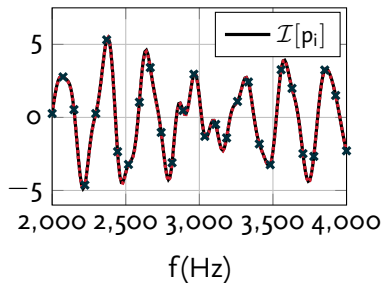
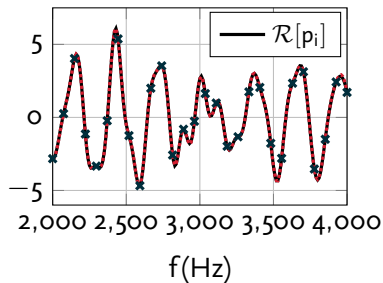


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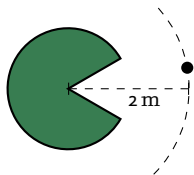
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- **Complex/real kernel + rational basis**



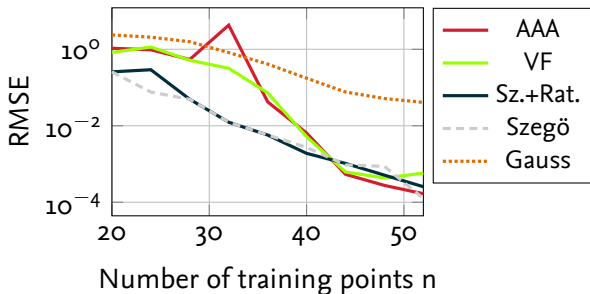
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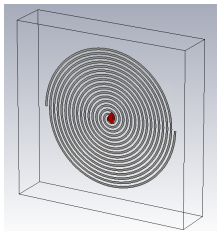


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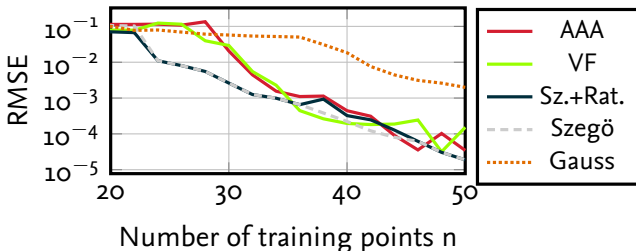
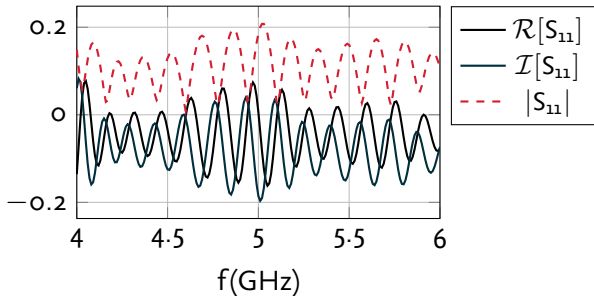


H. Ziegelwanger, P. Reiter, "The PAC-MAN model: Benchmark case for linear acoustics in

# Benchmark Problem: Sprial Antenna



CST Microwave Studio



# Conclusion

- Real/complex kernel interpolation (GPR)
- Construction of pseudo-kernels for FRF approximation
- Additional rational basis functions to capture dominant poles
- Model selection based on LOO and instability penalty
- Method is competitive w.r.t. AAA/VF for a number of benchmark problems
- GP tools for model selection and adaptive sampling are available

# Thank you for your attention!

N. Georg, J. Bect, U. Römer, S. Schöps. Rational kernel-based interpolation for complex-valued frequency response functions. Will appear on ARXIV.

**Acknowledgment:** This work is supported by the DFG grant RO4937/1-1, the Excellence Initiative of the German Federal and State Governments and the Graduate School of Computational Engineering at TU Darmstadt.