



HAL
open science

Some Remarks on the Implementation of a Derivative Action via a Delay-Difference Approximation

Diego Torres-García, César Fernando Méndez Barrios, Silviu-Iulian Niculescu

► **To cite this version:**

Diego Torres-García, César Fernando Méndez Barrios, Silviu-Iulian Niculescu. Some Remarks on the Implementation of a Derivative Action via a Delay-Difference Approximation. 2023 European Control Conference (ECC 2023), European Control Association; University Politehnica Bucharest, Jun 2023, Bucarest, Romania. hal-04130834

HAL Id: hal-04130834

<https://centralesupelec.hal.science/hal-04130834>

Submitted on 16 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some Remarks on the Implementation of a Derivative Action via a Delay-Difference Approximation

Diego Torres-García, César-Fernando Méndez-Barrios and Silviu-Iulian Niculescu

Abstract—The purpose of this note is to present the effects induced on the dynamics of a linear system using a derivative operator which is implemented by means of a delay-difference approximation. In this sense, a linear time-invariant (LTI) single-input/single-output (SISO) system is considered, subject to a PD-type controller for which the derivative action was implemented by considering the aforementioned mechanism. The use of such a methodology leads to a characteristic function with coefficients that explicitly depend on the *delay parameter*. This note presents sufficient conditions under which the approximation leads to unstable (real) roots, showing explicitly the sensitivity of the approximation with respect to “small” delays. Several illustrative examples complete the presentation.

I. INTRODUCTION

It is well-recognized that PID controllers are the preferred strategy implemented in industrial processes [1], [2]. Such *popularity* arises mainly from its capability to handle most of the control problems presented in the industry, as well as from its simplicity of design and implementation [3], [4]. However, as it is well-known, the inclusion of the derivative element in the control strategy can improve system performance, but only when such an element is applied correctly, otherwise, the derivative action may amplify noises or even cause instability [5]. To deal with such a problem, different approaches have been classically considered as, for example, adding a low-pass filter to the derivative action or approximating the derivative via the finite-difference method. This last tool is the main subject of this work; it is noteworthy that applying this technique leads to a characteristic function in which parameters explicitly depend on the delay term of the approximation.

It is worth mentioning that time-delay systems have been an object of study since the last century and are still of interest in a wide variety of applications (see, for example, [6]–[9]); however, studies that consider systems with parameters that explicitly depend on the delay are less frequently found (see [10] and [11], as examples).

In some situations, using a delay-difference approximation may induce instability in the system. The behavior of the unstable characteristic roots with respect to the delay parameter has already been addressed in the literature; see, for example, [5] and [12]. More precisely, if only one delay is used in the approximation scheme, one obtains the following controller (in frequency-domain):

$$C_\tau(s; \tau) = k_p + k_d \frac{(1 - e^{-\tau s})}{\tau}.$$

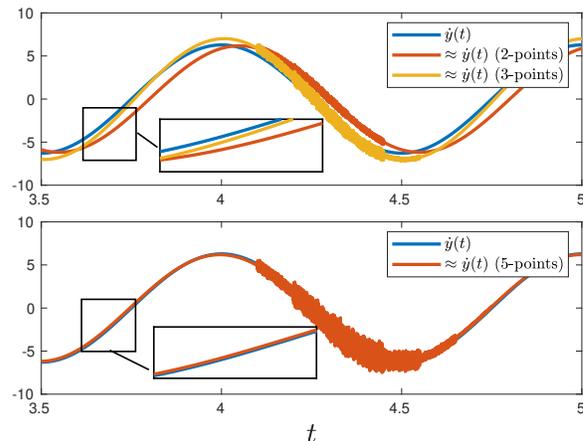


Fig. 1: Effect of adding more points to the derivative approximation.

In [13], an extension to the two delays case, namely

$$C_\tau(s; \tau) = k_p + k_d \frac{(3 - 4e^{-\tau s} + e^{-2\tau s})}{2\tau},$$

was proposed and discussed.

The present work aims to generalize these previous results to cases where an arbitrary number of commensurate delays is considered; the interest in generalizing the results mentioned above is twofold. On the one hand, the inclusion of more delays bears a better approximation to the pure derivative action. However, it is important to mention that as the number of delays increases, the noise amplification also increases. Therefore, it will depend on the application to decide the convenient number of delays to consider (see Figure 1 for an illustration of both properties). On the other hand, since increasing the number of delays corresponds to more degrees of freedom for the controller, it is interesting to investigate whether or not the well-behaved region changes when a more significant number of delays is considered. In other words, we may expect that some systems that cannot be stabilized by standard PD-controllers can be stabilized by using delay-difference approximations for implementing the derivative actions. Intuitively, such an idea makes sense since the delay-difference approximation makes the closed-loop system infinite-dimensional.

The remaining part of the paper is organized as follows: Section II presents the main results. A few illustrative examples are proposed in Section III and some concluding remarks end the paper. The notations throughout the paper are standard.

II. IMPROPERLY-POSED CASE CHARACTERIZATION

A. Problem Formulation

As mentioned in the Introduction, this work addresses LTI SISO systems with the following transfer function representation:

$$H(s) := \frac{P(s)}{Q(s)}, \quad (1)$$

where $P(s) := \sum_{i=0}^m p_i s^i$ and $Q(s) := s^n + \sum_{i=0}^{n-1} q_i s^i$ are polynomials in s with real coefficients, and $H(s)$ is strictly proper, that is $m := \deg(P) < \deg(Q) =: n$. It is assumed that a ‘‘classical’’ PD controller is used to control the system. Its frequency-domain representation writes as:

$$C(s) := k_p + k_d s, \quad (2)$$

where $(k_p, k_d) \in \mathbb{R} \setminus \{(0, 0)\}$ are the proportional and derivative gain, respectively. Our interest is to study the case where the derivative action is approximated via a *delay-difference approximation*, leading to the following frequency-domain controller:

$$C_\tau(s; \tau) := k_p + \frac{k_d}{\tau} \sum_{k=1}^{\ell} \frac{1}{k} (1 - e^{-\tau s})^k, \quad (3)$$

where $\tau > 0$ represents the delay of the approximation scheme and $\ell \geq 1$ represents the number of delays.

With the assumptions and notations above, the corresponding characteristic function of the closed-loop system when the delay-difference approximation is used writes as:

$$\Delta_\tau(s; \tau) := Q(s) + P(s)C_\tau(s; \tau). \quad (4)$$

This is known as a quasi-polynomial with delay-dependent coefficients and, as expected, it has an infinite number of roots [14], [15]. However, it is important to note that such solutions are still continuous functions of the system parameters and, in particular, this continuity holds for strictly positive delays τ .

Next, consider the following definition

Definition 1 (Improperly-posed system): Suppose a dynamical system represented by the transfer function (1) and a stabilizing controller $C(s)$ given by (2). The controller is called *improperly-posed* for ‘‘small’’ delays if after replacing it by $C_\tau(s; \tau)$ given by (3) with $\tau \rightarrow 0_+$ the closed-loop system becomes unstable, and it will be called *improperly-posed*. If after replacing the controller the closed-system remains stable it is called a properly-posed controller/closed-loop system.

B. Auxiliary time-delay system

As a first step, consider the following system:

$$\dot{y}(t) + \alpha \sum_{k=0}^{\ell} a_k y(t - k\tau) = 0 \quad (5)$$

$$\implies \Delta(s; \alpha) := s + \alpha \sum_{k=1}^{\ell} \frac{1}{k} (1 - e^{-s})^k,$$

where $\alpha \in \mathbb{R}$ is a parameter.

This system possesses the following properties: (i) The origin is always a root of the quasipolynomial (5). In particular, if $\alpha = -1$, such a root is double. (ii) the system will exhibit at least one solution with positive real part if $\alpha < -1$.

These properties, as shown in [5] (case $\ell = 1$) and [13] (case $\ell = 2$), play a crucial role when characterizing the roots of (4).

C. Main Results

In the sequel, we address the improperly-posed case given the closed-loop structure (1)-(3).

We have the following results:

Proposition 1: The closed-loop system $H-C_\tau$ given by (1)-(3) is improperly-posed for ‘‘small’’ delays if the quasipolynomial P_ℓ^α given by:

$$P_\ell^\alpha(w) = w + k_d p_m \sum_{k=1}^{\ell} \frac{1}{k} (1 - e^{-w})^k, \quad (6)$$

is unstable.

Proposition 2: The closed-loop system $H-C_\tau$ given by (1)-(3) is properly-posed for ‘‘small’’ delays if the relative degree of the system is greater than one. If the system has relative degree equal one and $P_\ell^\alpha(w)$ has unstable roots, then the system has at least one unstable solution $s^*(\tau) = z^*/\tau + \mathcal{O}(1)$, where z^* corresponds to an unstable root of $P_\ell^\alpha(w)$.

Proof Ideas: On the one hand, the structure of the solutions and the auxiliary quasi-polynomial can be found following the ideas presented in [16], where the solutions are expressed as a power series using the so-called Newton’s Diagram method; on the other hand, the behavior of the solution, and hence the conditions over $k_d p_m$, only need the properties of (5) presented in subsection II-B. In order to prove such properties, the structure of $\Delta(s; \alpha)$ has to be exploited following procedures similar to those presented in [5] and [13].

III. NUMERICAL EXAMPLES

The following examples aim to show the accuracy of the presented results and motivate future research in the area. All the solutions presented in the examples are computed using the QPmR algorithm; readers may refer to [17] for an in-deep explanation.

Auxiliary quasi-polynomial	Right-most roots
$P_3^\alpha(w) := w - 5 \left(\frac{11 - 18e^{-w} + 9e^{-2w} - 2^{-3w}}{6} \right)$	$\{0, -0.79 \pm 0.70j, 9.16\dots\}$
$P_4^\alpha(w) := w - 10 \left(\frac{25 - 48e^{-w} + 36e^{-2w} - 16e^{-3w} + 3e^{-4w}}{12} \right)$	$\{0, -0.55 \pm 1.05j, 20.83\dots\}$

TABLE I: Right-most roots of auxiliary quasi-polynomials from Example 3.1 and 3.2.

Example 3.1 (Gradient play dynamics): In this first example, the Gradient Play system is presented. Such a system has a first-order characteristic function; the intention is to show the effectiveness of the presented methodology when

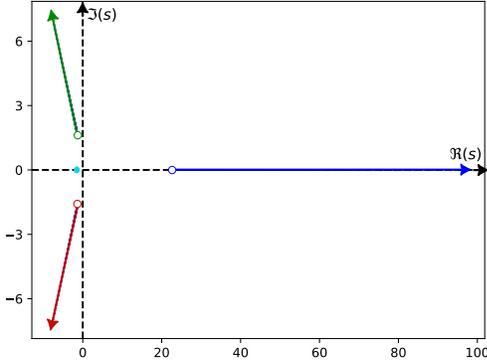


Fig. 2: Evolution of the system's roots as τ decreases its value from 0.5 to 0.1. As may be expected, it can be observed that the rightmost root moves to the right as τ becomes smaller.

computing the unstable root of an improperly-posed system. In this regard, consider the system presented in [18]; with $\lambda_i(J) = -5$ the characteristic equation reads:

$$\Delta(s) = s - 5(k_p + k_d s), \quad (7)$$

thus, $(k_p, k_d) = (1, 1)$ suffices to stabilize the system, placing a single root at $s = -5/4$. Observe that the relative degree of the systems equals one and that $k_d p_m = -5 < -1$, which indicated that the closed-loop system is improperly-posed. In fact, simple computations show that $s^*(\tau) = 9.16/\tau + \mathcal{O}(1)$ is a solution of the closed-loop system. Taking, for example, $\tau = 0.5$, the characteristic function presents a root at $s = 23.33$, which is close to the expected root $s^*(0.5) = 18.33 + \mathcal{O}(1)$. Figure 2 shows the behavior of the system's roots as the value of τ decreases. It is worth mentioning that the authors in [18] arrived to a similar conclusion by using a different argument.

Example 3.2 (Third-order dynamical system): This second example considers a third-order dynamical system with relative degree one; it intends to demonstrate the importance of computing higher-order terms of $s^*(\tau)$. In this vein, consider the system associated with the following transfer function:

$$H(s) = \frac{-5s^2}{s^3 + s^2 - s - 1}, \quad (8)$$

which is stable with roots at $s = \{-1.97, -0.013 \pm 0.23j\}$ when $(k_p, k_d) = (2, 2)$. Consider a derivative approximation with four delays, producing the auxiliary quasi-polynomial $P_4^a(w)$ (see Table I) for which $z^* = 20.83$ is a solution. With $\tau = 0.4$ the expected solution should be around $s^*(0.4) \approx 52.07$; computing the solutions of the characteristic equation, a solution at $s = 61.10$ is found.

Example 3.3 (Stabilizing effects): This last examples shows an advantage of considering a larger number of delays in the approximation. We consider a system

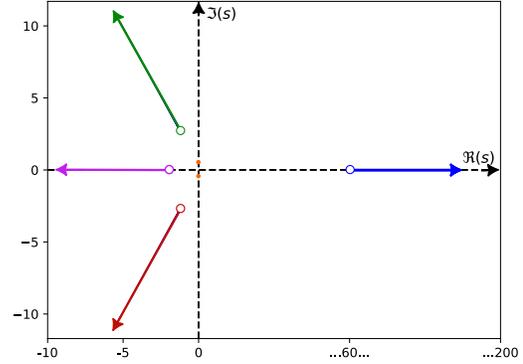


Fig. 3: Evolution of the system's roots as τ shifts from 0.4 to 0.1. Similar to the previous example, the rightmost root moves to the right as τ decreases.

described by the transfer function:

$$H(s) = \frac{1}{s^4 + s^2 + 4s - 31}. \quad (9)$$

Observe that the system's relative degree is greater than one; thus, the *improperly-posedness* property of the closed-loop system is not present; however, the absence of the third-order term bears a new problem: the system cannot be stabilized using a "classical" PD controller. Now, Figure 5 shows the systems response when $(k_p, k_d) = (31.4, -3.4)$. Here, it can be observed that the inclusion of a second-time delay in the approximation stabilizes the system. A similar result is presented in [19], where it is shown that in order to stabilize a chain of integrators using time delays, it is necessary to use at least n delays, where n is the number of integrators in the chain; the more the degrees of freedom there are, the more roots can be controlled. Indeed, for this example the delay τ represents a control parameter and has to be correctly selected in order to stabilize the system. It can be observed in Figure 4 that there is a value of τ for which some roots cross the axis from the right-half plane to the left-half plane, stabilizing the system.

IV. CONCLUDING REMARKS

This paper addressed the closed-loop stability of an LTI SISO system controlled by a PD controller where the derivative action is approximated by delay-difference operators including more than one delay. The case of commensurate delays in the approximation scheme was considered. Sufficient conditions for "properly-posed"/"improperly-posed" closed-loop systems were proposed. Such conditions depend on the relative degree of the transfer functions and on the behavior of the right-most characteristic root of an auxiliary first-order quasipolynomial. Several examples were performed to show both the presented technique's advantages as well as limitations.

From the numerical examples, future research should focus on computing higher-order terms of the solution $s^*(\tau)$. It is

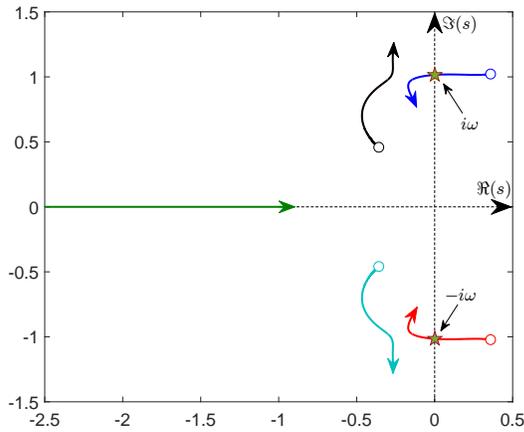


Fig. 4: Rightmost root behavior of the closed-loop system (8) for $\tau \in (0, 0.89525]$.

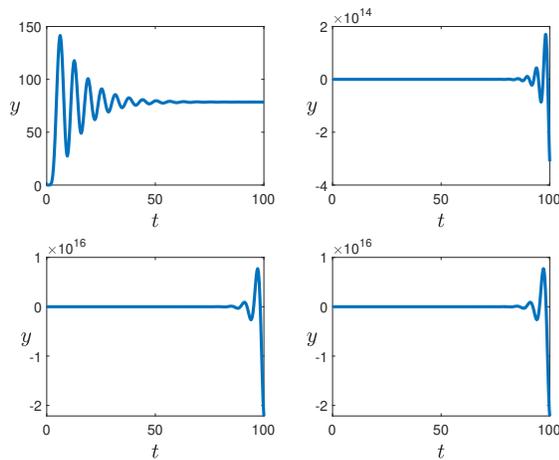


Fig. 5: System (9) temporal response to a step input when it is subjected to a PD controller with a derivative approximated with two delays (up left), one delay (upright), delay-free, i.e., pure derivative action (down left) and a filtered derivative action (down right).

also of interest to investigate the behavior of the solutions when the considered delays on the approximation are incommensurate or hold a different relation between them.

REFERENCES

- [1] I. Díaz-Rodríguez, S. Han, and S. Bhattacharyya, *Analytical Design of PID Controllers*. Springer Cham, 2019.
- [2] R. Borase, D. Maghade, S. Sondkar, and S. Pawar, “A review of PID control, tuning methods and applications,” *International Journal of Dynamics and Control*, vol. 9, no. 2, pp. 818–827, 2021.
- [3] K. J. Astrom and T. Haggglund, *PID controllers: theory, design, and tuning*, vol. 2. Instrument society of America Research Triangle Park, NC, 1995.
- [4] A. O’Dwyer, *Handbook of PI and PID Controller Tuning Rules*. London: Imperial College Press (ICP), 3rd ed., 2009.
- [5] C.-F. Méndez-Barrios, S.-I. Niculescu, A. Martínez-González, and A. Ramírez, “Characterizing some improperly posed problems in proportional-derivative control,” *International Journal of Robust and Nonlinear Control*, vol. 32, no. 18, pp. 9452–9474, 2022.
- [6] K. L. Cooke and Z. Grossman, “Discrete delay, distributed delay and stability switches,” *Journal of mathematical analysis and applications*, vol. 86, no. 2, pp. 592–627, 1982.

- [7] G. J. Silva, A. Datta, and S. Bhattacharyya, “Pi stabilization of first-order systems with time delay,” *Automatica*, vol. 37, no. 12, pp. 2025–2031, 2001.
- [8] R. Sipahi, S.-I. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu, “Stability and stabilization of systems with time delay,” *IEEE Control Systems Magazine*, vol. 31, pp. 38–65, Feb 2011.
- [9] A. Ramirez, S. Mondié, R. Garrido, and R. Sipahi, “Design of proportional-integral-retarded (pir) controllers for second-order lti systems,” *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1688–1693, 2015.
- [10] C. Jin, K. Gu, S.-I. Niculescu, and I. Boussaada, “Stability analysis of systems with delay-dependent coefficients subject to some particular delay structure,” in *2018 European Control Conference (ECC)*, pp. 1206–1213, IEEE, 2018.
- [11] G. Lai, S. Liang, and Z. Wang, “Algorithms for fast calculation of spectral abscissa for retarded time-delay systems with delay-dependent coefficients,” *IFAC-PapersOnLine*, vol. 54, no. 18, pp. 102–107, 2021.
- [12] A. Martínez-González, C.-F. Méndez-Barrios, S.-I. Niculescu, and J. Chen, “Insights on asymptotic behavior of characteristic roots of quasi-polynomials with delay-dependent coefficients in some ill-posed cases,” *IFAC-PapersOnLine*, vol. 52, no. 18, pp. 73–78, 2019.
- [13] D. Torres-García, C.-F. Méndez-Barrios, and S.-I. Niculescu, “Delay-difference approximation of pd-controllers. insights into improperly-posed closed-loop systems,” in *Proceedings of the 1st IFAC Workshop on Control of Complex Systems*, IFAC, 2022.
- [14] R. Bellman and K. L. Cooke, *Differential-difference equations*. New York: Academic Press, 1963.
- [15] W. Michiels and S.-I. Niculescu, *Stability, Control, and Computation for Time-Delay Systems. An Eigenvalue-Based Approach (2. ed.)*. Advances in design and control, Philadelphia: SIAM, 2014.
- [16] A. Martínez-González, C.-F. Méndez-Barrios, S.-I. Niculescu, J. Chen, and L. Félix, “Weierstrass approach to asymptotic behavior characterization of critical imaginary roots for retarded differential equations,” *SIAM Journal on Control and Optimization*, vol. 57, no. 1, pp. 1–22, 2019.
- [17] T. Vyhřídál and P. Zitek, “Qpmr-quasi-polynomial root-finder: Algorithm update and examples,” in *Delay systems*, pp. 299–312, Springer, 2014.
- [18] R. Sipahi, G. Arslan, and S.-I. Niculescu, “Some remarks on control strategies for continuous gradient play dynamics,” in *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 1966–1971, IEEE, 2006.
- [19] S.-I. Niculescu and W. Michiels, “Stabilizing a chain of integrators using multiple delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 802–807, 2004.