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# Scalable Stability of Nonlinear Interconnected Systems in case of amplifying Perturbations

Marco Mirabilio, *Member, IEEE*, Alessio Iovine, *Member, IEEE*

**Abstract**—This paper investigates the stability of large-scale systems (LSSs) in the presence of subsystems that amplify the perturbations propagated by their neighborhood, possibly leading to undesired behaviors of the overall interconnected system. Then, sufficient conditions ensuring the system trajectories boundedness and the subsequent LSS asymptotic stability in the sense of scalable Mesh Stability are proven to exist. The theoretical results show that there exists a dependence between the stability and the topology of the interconnected system. The obtained framework is then exploited for the stability analysis of a network of electrical microgrids, showing the effectiveness of the theoretical results.

**Index Terms** - scalable Mesh Stability, large-scale systems, nonlinear interconnected systems stability, Input-to-State Stability

## I. INTRODUCTION

The purpose of this work is to investigate the stability of large-scale systems (LSSs) with general network topology (or interconnection graph), where the subsystems (or agents) are affected by external disturbances. The family of LSSs has received considerable attention through the years since their relevance in several fields [1]. Indeed, technological improvements are leading to an ever increasing complexity of systems, often composed of many interconnected subsystems [1], [2]. Unfortunately, the size and the complexity of LSSs make their stability investigation a difficult task. For this reason, decompositional approaches based on breaking the LSS into subsystems with lower complexity are preferred [3], [4], [5]. However, the problem of assessing the overall LSS stability on the basis of local properties arises. In this regard, the notion of Input-to-State Stability (ISS, see [6]) has been proven to provide suitable tools for investigating the stability of interconnected systems [7]. The present work lies within the family of ISS-based approaches for LSS stability investigation. The objective is to provide stability guarantees in presence of perturbations that are amplified inside the network. We target a result that is scalable, i.e., the LSS stability can be verified by local conditions on clusters of agents independently on their growth in number. In particular, it is based on the framework introduced in [8], and extends it to the case of interconnected systems with agents amplifying the perturbations propagated by their

neighbors. Then, the implications about the scalability of the result and the need for the knowledge of the LSS interconnection graph are discussed.

In [8], the authors consider a LSS affected by external disturbances, without any constraint on the interconnection topology. With the purpose of extending the notion of String Stability for vehicular platoons (see [9], [10]) to general LSSs, in [8] the notion of scalable Mesh Stability (sMS) is introduced to describe a system that is able to dampen the perturbations propagating through it. To this purpose, and by exploiting a “trajectory-based” approach (namely  $\mathcal{K}_\infty, \mathcal{KL}$  functions), sufficient conditions to be met on the interconnection function of each subsystem are proven to exist, thus ensuring the LSS to be sMS. Differently from here, in [8] all the subsystems are supposed to have ideal interconnection links, thus to be able to dampen the perturbations propagated by their neighbors. The definition of sMS follows from the notion of scalable ISS introduced in [4], where the authors consider a LSS composed by agents with same dynamics, affected by external disturbances, and with neighborhood set having the same cardinality, leading to a symmetric interconnection topology. With the objective of providing a scalable notion of network performance, sufficient conditions leading to an ISS test to be performed on the agents of the LSS are proven to exist by means of a “Lyapunov-based” approach. In this regard, the uniformity of the LSS plays a central role in proving its ISS property by means of a max-separable Lyapunov function. Differently, in this paper, we consider a more general framework than the ones in [8], [4] and [11]: the subsystems may have different dynamics and no pre-defined neighborhood set. This approach has similar foundations to the one in [12], where time-delays are considered but the strong requirement of asymptotic stability for each agent is necessary, contrarily from what considered here. Other examples of ISS-based approaches are shown in [7] and [13], where compositions of ISS subsystems are considered. In these works, the interconnection between the agents is supposed to be described by nonlinear gains. Then, grouping the interconnection functions in a gain matrix and exploiting the asymptotic gain property, the existence of a small-gain condition that ensures the ISS of the overall interconnected system is proven. However, no scalability properties are provided, hence the existence of trajectory bounds that do not depend on the number of subsystems.

In this work, we consider a LSS with agents causing local amplification of the perturbations propagating through it. Inspired by the classical Lyapunov stability classification (see [14]), and similarly to [13], we first introduce the

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definition of scalable Marginal Mesh Stability (sMMS). As sMS, sMMS requires the existence of trajectory bounds that do not depend on the number of subsystems. However, it ensures only that the system trajectories are bounded, instead of their asymptotic convergence to the equilibrium. Subsequently, we investigate the existence of sufficient conditions for the sMMS of the LSS. Then, for the same framework, we show how considering stronger assumptions on the local stability of each agent leads to deriving sMS, thus obtaining information about the asymptotic behavior of the system.

The proven sufficient conditions, both sMMS or sMS, ensure stability for the whole LSS, if verified by each subsystem. Therefore, they represent a local test that implies global results. The consequence for the LSS being either sMMS or sMS, or both, is that the effects of the amplified perturbations remain local, avoiding the rising of undesired behaviors inside the system. The results show that for the analysis of agents amplifying the propagating perturbations we need to investigate the capability of their neighbors to counterbalance the amplified perturbations. This leads to the necessity of further information about the local sections of the LSS in which these agents are located. In addition, the results show how the stability region enlarges as the number of subsystems needed to counteract the effects of the unstable ones increases. However, this dependence remains bounded within the borders of such a stable neighborhood, ensuring the existence of trajectory bounds that do not depend on the total number of agents in the LSS, thus their scalability with respect to the interconnected system size. The obtained results are general and include the special case shown in [8] with stronger constraints on the interconnection links. Finally, the theoretical analysis is exploited to investigate numerically the stability of an interconnected system representing an aggregate of microgrids composed by two different kinds of nodes.

The rest of the paper is organized as follows. Section II contains graph-related notation and definitions, and introduces the considered framework. Section III contains the main result about stability of LSS. Section IV describes a numerical example. Section V provides conclusive remarks.

**Notation** -  $\mathbb{R}_{\geq 0}$  is the set of non-negative real numbers. For a vector  $x \in \mathbb{R}^n$ ,  $|x| = \sqrt{x^T x}$  is its Euclidean norm,  $|x|_\infty = \max_{i=1 \dots n} |x_i|$  is its infinity norm and  $|x(\cdot)|_\infty^{[t_0, t]} = \sup_{t_0 \leq \tau \leq t} |x(\tau)|$  is the  $\mathcal{L}_\infty$  signal norm. Symbol  $\lceil \cdot \rceil$  denotes the ceiling function, and  $|\cdot|$  denotes the cardinality of the set in argument. We refer to [14] and [15] for the definition of Lyapunov functions and of class  $\mathcal{K}$ ,  $\mathcal{K}_\infty$ ,  $\mathcal{KL}$  functions.

## II. PRELIMINARIES

### A. Graph notation

In this section, we recall graph related notation and definitions from [16] and [17]. Let  $V = \{v_1, v_2, \dots, v_{N_V}\}$  be a set of cardinality  $N_V$ . We denote with  $[V]^2$  the set of 2-elements subsets of  $V$ .

*Definition 1:* A *directed graph* (or *digraph*) is a pair  $\mathcal{D} = (V, E)$ , where  $V$  is the set of vertices, and  $E \subseteq$

$\{(x, y) | (x, y) \in [V]^2\}$  is the set of edges made by ordered pairs of vertices, thus having an orientation.

For further use,  $V(\mathcal{D})$  and  $E(\mathcal{D})$  refer to vertices and edges of  $\mathcal{D}$ , respectively. If an edge exists between two vertices  $x, y \in V$ , with direction  $x \rightarrow y$ , we denote it with  $e_x^y = (x, y)$ .

*Definition 2:* The transpose of a digraph  $\mathcal{D}$  is denoted with  $\mathcal{D}^T$ , where  $V(\mathcal{D}^T) = V(\mathcal{D})$ , but with all of the edges reversed compared to the orientation of the corresponding edges in  $\mathcal{D}$ .

*Definition 3:* Let  $\mathcal{D}$  be a digraph. A *directed trail* is a finite or infinite sequence  $e_0^1, e_1^2, \dots, e_{k-1}^k$  of distinct edges in  $E(\mathcal{D})$  joining a sequence of vertices  $v_0, v_1, \dots, v_k$  in  $V(\mathcal{D})$ . The length of the trail is equal to the number of edges in the sequence.

For future use, we define function  $L : E(\mathcal{D}) \times \dots \times E(\mathcal{D}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  that given a trail  $T$  returns its length  $L(T)$ .

*Definition 4:* Let  $T_1 = e_0^1, e_1^2, \dots, e_{k-1}^k$  and  $T_2 = \bar{e}_0^1, \bar{e}_1^2, \dots, \bar{e}_{l-1}^l$  be two trails in  $\mathcal{D}$ , with  $k \leq l$ . If  $T_2$  is such that  $T_2 = \bar{e}_0^1, \bar{e}_1^2, \dots, e_0^1, e_1^2, \dots, e_{k-1}^k, \dots, \bar{e}_{l-1}^l$ , then  $T_1$  is said to be *contained* in  $T_2$ . In this case, we also refer to  $T_1$  as a *subtrail* of  $T_2$ . If a trail is non contained in a longer one, it is said *maximal*.

### B. Large-Scale System framework

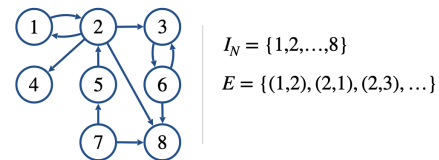


Fig. 1: Example of interconnected system described by a digraph  $\mathcal{D} = \{\mathcal{I}_N, E\}$ .

As in [8], we consider a LSS composed of  $N \in \mathbb{N}$  interconnected subsystems, with dynamics

$$\dot{x}_i = f_i(x_i, \{x_j\}_{j \in \mathcal{N}_i}, d_i), \quad i \in \mathcal{I}_N, \quad (1)$$

where  $\mathcal{I}_N = \{1, \dots, N\}$ ,  $x_i \in \mathbb{R}^{n_i}$ ,  $n_i \in \mathbb{N}$ , is the state vector,  $d_i \in \mathbb{R}^{m_i}$ ,  $m_i \in \mathbb{N}$ , is the disturbance,  $\mathcal{N}_i \subseteq \mathcal{I}_N$  denotes the neighbors set, and  $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_{i,1}} \dots \times \mathbb{R}^{n_{i,|\mathcal{N}_i|}} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$  is the vector field associated to the  $i$ -th subsystem (also called agent). Symbol  $n_{i,j}$  denotes the state dimension of the  $j$ -th neighbor of  $i$ . We assume that  $f_i(0, 0, \dots, 0, 0) = 0$ ,  $\forall i \in \mathcal{I}_N$ , and that the interconnected system (1) is forward complete, meaning that its solution exists for all initial states, and all disturbances  $d_i$ . Moreover, we assume that the interconnections in (1) are described by a directed graph  $\mathcal{D} = (\mathcal{I}_N, E)$  (for example, see Fig. 1). Then, neighborhood of agent  $i \in \mathcal{I}_N$  is defined as  $\mathcal{N}_i = \{j \in \mathcal{I}_N \mid \exists e_j^i \in E\}$ . For the problem not to be trivial, graph  $\mathcal{D}$  is assumed to be weakly connected.

We introduce here the following definition:

*Definition 5:* The system (1) is said to be *scalable Marginally Mesh Stable* (sMMS) if there exist  $\sigma_i^x, \sigma_i^d, \gamma_i \in$

$\mathcal{K}_\infty$  such that the stability property

$$|x_i(t)| \leq \sigma_i^x (|x_i(t_0)|) + \gamma_i \left( \max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[t_0, t]} \right) + \sigma_i^d \left( |d_i(\cdot)|_\infty^{[t_0, t]} \right), \quad t \geq t_0 \geq 0, \quad (2)$$

of each  $i \in \mathcal{I}_N$ ,  $\forall N \in \mathbb{N}$ , implies the existence of some functions  $\sigma^x, \sigma^d \in \mathcal{K}_\infty$  such that, for each  $t \geq t_0$ ,

$$\max_{i \in \mathcal{I}_N} |x_i(t)| \leq \sigma^x \left( \max_{i \in \mathcal{I}_N} |x_i(t_0)| \right) + \sigma^d \left( \max_{i \in \mathcal{I}_N} |d_i(\cdot)|_\infty^{[t_0, t]} \right), \quad (3)$$

for any initial condition  $x_i(t_0) \in \mathbb{R}^n$  and any disturbance function  $d_i(\cdot)$ .

For sake of completeness, and to better remark the differences with respect to the concept of sMMS, we recall here the definition of scalable Mesh Stability introduced in [8]:

*Definition 6:* The system (1) is said to be *scalable Mesh Stable* (sMS) if there exist  $\beta_i^x \in \mathcal{K}\mathcal{L}$  and  $\gamma_i, \sigma_i^d \in \mathcal{K}_\infty$  such that the ISS property

$$|x_i(t)| \leq \beta_i^x (|x_i(t_0)|, t - t_0) + \gamma_i \left( \max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[t_0, t]} \right) + \sigma_i^d \left( |d_i(\cdot)|_\infty^{[t_0, t]} \right), \quad t \geq t_0 \geq 0 \quad (4)$$

of each  $i \in \mathcal{I}_N$ ,  $\forall N \in \mathbb{N}$ , implies the existence of some functions  $\beta^x \in \mathcal{K}\mathcal{L}$  and  $\sigma^d \in \mathcal{K}_\infty$  such that, for each  $t \geq t_0$ ,

$$\max_{i \in \mathcal{I}_N} |x_i(t)| \leq \beta^x \left( \max_{i \in \mathcal{I}_N} |x_i(t_0)|, t \right) + \sigma^d \left( \max_{i \in \mathcal{I}_N} |d_i(\cdot)|_\infty^{[t_0, t]} \right), \quad (5)$$

for any initial condition  $x_i(t_0) \in \mathbb{R}^n$  and any disturbance function  $d_i(\cdot)$ .

### III. MAIN RESULTS

For sake of simplicity, here we refer to the transpose digraph  $\mathcal{D}^T$  describing the interconnection topology of (1). In this case, set  $\mathcal{N}_i$  corresponds to all the vertices in  $\mathcal{I}_N$  for which it exists  $e_i^j \in E(\mathcal{D}^T)$ . Moreover, let  $\{T_i^h\}_{h=1}^{\bar{T}_i}$  be the set of maximal trails in  $\mathcal{D}^T$  with starting vertex  $i \in \mathcal{I}_N$  (see Definition 4), where  $\bar{T}_i$  is the cardinality of the set, since for each node, several maximal trails could exist. Then, the following assumptions are made:

*Assumption 1:* For each  $i \in \mathcal{I}_N$  for which either (2) or (4) hold, there exists  $\tilde{\gamma}_i > 0$  such that  $\gamma_i(s) \leq \tilde{\gamma}_i s$ ,  $\forall s \geq 0$ .

*Assumption 2:* For each  $i \in \mathcal{I}_N$ , for each maximal trail in  $\{T_i^h\}_{h=1}^{\bar{T}_i}$ , there exists a trail  $\tilde{T}_i^h$  contained in  $T_i^h$ , with initial vertex  $i$  and a finite length  $L(\tilde{T}_i^h) \leq L(T_i^h)$ , such that the vertices  $i, i_1, \dots, i_{L(\tilde{T}_i^h)}$  corresponding to the sequence of edges  $e_i^{i_1}, e_{i_1}^{i_2}, \dots, e_{i_{L(\tilde{T}_i^h)-1}}^{i_{L(\tilde{T}_i^h)}}$  in  $\tilde{T}_i^h$  verify  $\prod_{v=0}^{L(\tilde{T}_i^h)-1} \tilde{\gamma}_{i_v} < 1$ , where  $i_0 = i$ , thus  $\tilde{\gamma}_{i_0} = \tilde{\gamma}_i$ .

For future use, we introduce  $\hat{\sigma}^p = \max_{i \in \mathcal{I}_N} \sigma_i^p(s)$ ,  $p \in \{x, d\}$ , and  $\hat{\beta}(r, s) = \max_{i \in \mathcal{I}_N} \beta_i^x(r, s)$ ,  $r, s \geq 0$ . It is easily seen that  $\hat{\sigma}^x, \hat{\sigma}^d \in \mathcal{K}_\infty$  and  $\hat{\beta} \in \mathcal{K}\mathcal{L}$ . Moreover, let  $t_1, t_2, \bar{t}$  be fixed time instants in  $[t_0, t]$ ; then we define  $|x(\bar{t})|_\infty = \max_{i \in \mathcal{I}_N} |x_i(\bar{t})|$  and  $d_\infty^{[t_1, t_2]} = \max_{i \in \mathcal{I}_N} |d_i(\cdot)|_\infty^{[t_1, t_2]}$ .

We now introduce the main results of the paper:

*Theorem 1:* Consider the LSS (1). Assume that (2) holds for any  $i \in \mathcal{I}_N$ . If Assumptions 1 and 2 hold for all the subsystems, then the interconnected system (1) is sMMS.

*Proof:* Theorem 1 can be proven by following similar arguments as in [8, Theorem 1]. By iteratively running through all possible trails starting from each node  $i$  and by Assumption 2, every trail has a sub-trail such that  $\prod_{v=0}^{L(\tilde{T}_i^h)-1} \tilde{\gamma}_{i_v} < 1$  is verified. Then, since also the worst-case equivalent interconnection parameter is lower than 1, the sMMS of the entire interconnected system is proven. ■

Theorem 1 extends the results in [8] by ensuring the boundedness of the overall system trajectories in the case of relaxed conditions. Indeed, differently from [8, Theorem 1], weaker conditions on the local stability of the subsystems are required, that is the existence of class  $\mathcal{K}_\infty$  functions instead of class  $\mathcal{K}\mathcal{L}$  ones, as well as the relaxation of the condition  $\tilde{\gamma}_i \in (0, 1)$  for all  $i$ , now  $\tilde{\gamma}_i > 0$ . However, we stress that the case  $\tilde{\gamma}_i \in (0, 1)$  turns out to be a special case of the general one considered in Theorem 1. In order to prove that the above result is general and that can lead to the SMS of the interconnected system, we introduce the following theorem requiring stronger conditions on the stability of each subsystem:

*Theorem 2:* Consider the LSS (1). Assume that (4) holds for any  $i \in \mathcal{I}_N$ . If Assumptions 1 and 2 hold for all the subsystems, then the interconnected system (1) is SMS.

*Proof:* Let be  $\tau \in [t_0, t]$ . For each  $i \in \mathcal{I}_N$ , by (4) with  $t_0 = \tau$ , we get:

$$|x_i(t)| \leq \hat{\beta}(|x_i(\tau)|, t - \tau) + \tilde{\gamma}_i \max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[\tau, t]} + \hat{\sigma}^d \left( |d_i(\cdot)|_\infty^{[\tau, t]} \right). \quad (6)$$

Given  $\bar{\omega} \in (0, 1)$ , for any  $t > t_0 > 0$  define  $M = \hat{M}[t - t_0]$ ,  $\hat{M} > 0$  arbitrarily large, and  $\tau_k = t_0 + \bar{\omega}^{M-k+1}(t - t_0)$ . By construction, sequence  $\{\tau_k\}_{k=0}^M$  is increasing and  $\tau_k \in (t_0, t)$ . Then, setting  $\tau = \tau_M$  in (6) we get:

$$|x_i(t)| \leq \hat{\beta}(|x_i(\tau_M)|, t - \tau_M) + \tilde{\gamma}_i \max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[\tau_M, t]} + \hat{\sigma}^d \left( |d_\infty^{[t_0, t]} \right), \quad (7)$$

where  $t - \tau_M = (1 - \bar{\omega})(t - t_0)$ . At this point, a bound for  $\max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[\tau_M, t]}$  needs to be derived. For this purpose, let us consider the trajectories at time instant  $\tau_M$  with initial condition in  $\tau = \tau_{M-1}$ . Let  $i_1 = \arg \max_{j \in \mathcal{N}_i} |x_j(\cdot)|_\infty^{[\tau_M, t]}$ .

Then, from (6) we get:

$$|x_{i_1}(\cdot)|_\infty^{[\tau_M, t]} \leq \hat{\beta}(|x_{i_1}(\tau_{M-1})|, \tau_M - \tau_{M-1}) + \tilde{\gamma}_{i_1} \max_{j \in \mathcal{N}_{i_1}} |x_j(\cdot)|_\infty^{[\tau_{M-1}, \tau_M]} + \hat{\sigma}^d \left( |d_\infty^{[t_0, t]} \right), \quad (8)$$

where  $\tau_M - \tau_{M-1} = (1 - \bar{\omega})\bar{\omega}(t - t_0)$ . Since by [15, Lemma 7] there exist  $\lambda_i \in \mathbb{R}_{>0}$  and  $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$  such that  $\beta_i^x(r, s) \leq \alpha_{1,i}^{-1}(\alpha_{2,i}(r)e^{-\lambda_i s})$ , then, by setting  $s = 0$  and defining  $\sigma_i^x(r) = \alpha_{1,i}^{-1}(\alpha_{2,i}(r)) \in \mathcal{K}_\infty$ , the sMMS of (1) follows from Theorem 1. Then,  $|x_i(t)| \leq A$ , for each

$i \in \mathcal{I}_N$ , where  $A = \sigma^x(|x(t_0)|_\infty) + \sigma^d(d_\infty^{[t_0, t]})$ . Therefore, by exploiting the sMMS property and (8) in (7), we get

$$|x_i(t)| \leq \sum_{k=-1}^0 \prod_{v=-1}^k \tilde{\gamma}_{i_v} \hat{\beta}(A, (1-\bar{\omega})\bar{\omega}^{k+1}(t-t_0)) \\ + \sum_{k=-1}^0 \prod_{v=-1}^k \tilde{\gamma}_{i_v} \hat{\sigma}^d(d_\infty^{[t_0, t]}) + \prod_{v=0}^1 \tilde{\gamma}_{i_v} \max_{j \in \mathcal{N}_{i_1}} |x_j(\cdot)|_\infty^{[\tau_{M-1}, \tau_M]}$$

where  $\tilde{\gamma}_{i_{-1}} = 1$  and  $\tilde{\gamma}_{i_0} = \tilde{\gamma}_i$ . Since Assumption 2 holds, for each sequence of vertices  $j_v = \arg \max_{j \in \mathcal{N}_{j_{v-1}}} |x_j(\cdot)|_\infty^{[t_0, t]}$  there exists a finite number  $l$  such that  $\prod_{v=0}^l \tilde{\gamma}_{i_v} < 1$ ,  $\tilde{\gamma}_{i_0} = \tilde{\gamma}_i$ . Then, by repeating the previous steps for the entire sequence of length  $l$ , we get

$$|x_i(t)| \leq \sum_{k=-1}^{l-1} \prod_{v=-1}^k \tilde{\gamma}_{i_v} \hat{\beta}(A, (1-\bar{\omega})\bar{\omega}^{k+1}(t-t_0)) \\ + \hat{c}_i^{l,d} \hat{\sigma}^d(d_\infty^{[t_0, t]}) + \hat{\gamma}_i^l \max_{j \in \mathcal{N}_{i_v}} |x_j(\cdot)|_\infty^{[\tau_{M-l}, \tau_M]} \quad (9)$$

where  $\hat{c}_i^{l,d} = \sum_{k=-1}^{l-1} \prod_{v=-1}^k \tilde{\gamma}_{i_v}$  and  $\hat{\gamma}_i^l = \prod_{v=0}^l \tilde{\gamma}_{i_v} < 1$ . As in [18], a  $\mathcal{KL}$  function  $\phi$  is defined as

$$\phi(r, s) = \sup_{\omega \in (0,1)} \omega^q \hat{\beta}(r, \omega s), \quad (10)$$

for some  $q > 0$ . Then, multiplying and dividing  $\hat{\beta}$  in (9) for  $((1-\bar{\omega})\bar{\omega}^k)^q$ , we get

$$|x_i(t)| \leq \hat{c}_i^{l,x}(\bar{\omega}) \phi(A, t-t_0) + \hat{c}_i^{l,d} \hat{\sigma}^d(d_\infty^{[t_0, t]}) \\ + \hat{\gamma}_i^l \max_{j \in \mathcal{I}_N} |x_j(\cdot)|_\infty^{[\tau_{M-l}, \tau_M]}$$

where  $\hat{c}_i^{l,x}(\bar{\omega}) = \frac{1}{(1-\bar{\omega})^q} \sum_{k=-1}^{l-1} \bar{\omega}^{-kq} \left( \prod_{v=-1}^k \tilde{\gamma}_{i_v} \right)$ . By Assumption 2, the same arguments apply for all the trails in  $\{T_i^h\}_{h=1}^{\bar{T}_i}$ . Then, let

$$\tilde{T}_i^x = \arg \max_{\substack{\bar{T}_i^h \\ h=1, \dots, \bar{T}_i}} \hat{c}_i^{L(\bar{T}_i^h), x}(\bar{\omega}), \quad \tilde{T}_i^d = \arg \max_{\substack{\bar{T}_i^h \\ h=1, \dots, \bar{T}_i}} \hat{c}_i^{L(\bar{T}_i^h), d}, \\ \tilde{T}_i^\gamma = \arg \max_{\substack{\bar{T}_i^h \\ h=1, \dots, \bar{T}_i}} \tilde{\gamma}_i^{L(\bar{T}_i^h)},$$

we define  $\hat{c}_i^x(\bar{\omega}) = \hat{c}_i^{L(\tilde{T}_i^x), x}(\bar{\omega})$ ,  $\hat{c}_i^d = \hat{c}_i^{L(\tilde{T}_i^d), d}$ ,  $\hat{\gamma}_i = \tilde{\gamma}_i^{L(\tilde{T}_i^\gamma)}$ , and  $\bar{L}_i = \max(L(\tilde{T}_i^x), L(\tilde{T}_i^d), L(\tilde{T}_i^\gamma))$ . For each  $i \in \mathcal{I}_N$ , we get

$$|x_i(t)| \leq \hat{c}_i^x(\bar{\omega}) \phi(A, t-t_0) + \hat{c}_i^d \hat{\sigma}^d(d_\infty^{[t_0, t]}) \\ + \hat{\gamma}_i \max_{j \in \mathcal{I}_N} |x_j(\cdot)|_\infty^{[\tau_{M-\bar{L}_i}, t]}. \quad (11)$$

By defining  $\hat{c}^x(\bar{\omega}) = \max_{i \in \mathcal{I}_N} \hat{c}_i^x(\bar{\omega})$ ,  $\hat{c}^d = \max_{i \in \mathcal{I}_N} \hat{c}_i^d$ ,  $\hat{\gamma} = \max_{i \in \mathcal{I}_N} \hat{\gamma}_i$ , and  $\hat{L} = \max_{i \in \mathcal{I}_N} \bar{L}_i$ , we get

$$|x_i(t)| \leq \hat{c}^x(\bar{\omega}) \phi(A, t-t_0) + \hat{c}^d \hat{\sigma}^d(d_\infty^{[t_0, t]}) \\ + \hat{\gamma} \max_{j \in \mathcal{I}_N} |x_j(\cdot)|_\infty^{[\tau_{M-\hat{L}}, t]}. \quad (12)$$

At this point, we can repeat the same steps for  $\tilde{j} = \arg \max_{j \in \mathcal{I}_N} |x_j(\cdot)|_\infty^{[\tau_{M-\hat{L}}, t]}$ . Then, there exists a sequence

of edges with length  $l$  such that the associated vertices lead to

$$|x_{\tilde{j}}(\cdot)|_\infty^{[\tau_{M-\hat{L}}, t]} \leq \sum_{k=-1}^{l-1} \prod_{v=-1}^k \tilde{\gamma}_{i_v} \hat{\beta}(A, (1-\bar{\omega})\bar{\omega}^{k+\hat{L}+1}(t-t_0)) \\ + \hat{c}_i^{l,d} \hat{\sigma}^d(d_\infty^{[t_0, t]}) + \hat{\gamma}_i^l \max_{j \in \mathcal{N}_{i_v}} |x_j(\cdot)|_\infty^{[\tau_{M-l-\hat{L}}, t]}. \quad (13)$$

By the straightforward calculations seen before,

$$|x_{\tilde{j}}(\cdot)|_\infty^{[\tau_{M-\hat{L}}, t]} \leq \hat{c}^x(\bar{\omega}) \bar{\omega}^{-\hat{L}q} \phi(A, t-t_0) + \hat{c}^d \hat{\sigma}^d(d_\infty^{[t_0, t]}) \\ + \hat{\gamma} \max_{j \in \mathcal{I}_N} |x_j(\cdot)|_\infty^{[\tau_{M-2\hat{L}}, t]}. \quad (14)$$

Using (14) in (12), we get

$$|x_i(t)| \leq \sum_{\xi=0}^1 \left( \hat{\gamma} \bar{\omega}^{-\hat{L}q} \right)^\xi \hat{c}^x(\bar{\omega}) \phi(A, t-t_0) \\ + \sum_{\xi=0}^1 \hat{\gamma}^\xi \hat{c}^d \hat{\sigma}^d(d_\infty^{[t_0, t]}) + \hat{\gamma} A. \quad (15)$$

By choosing  $\hat{M} = \hat{L}$ , defining  $\Xi = \frac{M}{\hat{L}} - 1 = \lceil t-t_0 \rceil - 1$ , and by repeating the steps leading to inequality (15), we get

$$|x_i(t)| \leq \sum_{\xi=0}^{\Xi} \left( \hat{\gamma} \bar{\omega}^{-\hat{L}q} \right)^\xi \hat{c}^x(\bar{\omega}) \phi(A, t-t_0) \\ + \sum_{\xi=0}^{\Xi} \hat{\gamma}^\xi \hat{c}^d \hat{\sigma}^d(d_\infty^{[t_0, t]}) + \hat{\gamma}^{\Xi+1} A. \quad (16)$$

If  $\hat{\gamma} \bar{\omega}^{-\hat{L}q} < 1$ ,

$$|x_i(t)| \leq \frac{\hat{c}^x(\bar{\omega})}{1 - \hat{\gamma} \bar{\omega}^{-\hat{L}q}} \phi(A, t-t_0) \\ + \hat{\gamma}^{\Xi+1} A + \frac{\hat{c}^d}{1 - \hat{\gamma}} \hat{\sigma}^d(d_\infty^{[t_0, t]}). \quad (17)$$

By defining  $\bar{\beta}(r, s) = \frac{\hat{c}^x(\bar{\omega})}{1 - \hat{\gamma} \bar{\omega}^{-\hat{L}q}} \phi(r, s) \in \mathcal{KL}$  and  $\bar{\sigma}(s) = \frac{\hat{c}^d}{1 - \hat{\gamma}} \hat{\sigma}^d(s)$ , we get

$$|x_i(t)| \leq \bar{\beta}(A, t-t_0) + \bar{\sigma}(d_\infty^{[t_0, t]}). \quad (18)$$

By recalling that  $A = \sigma^x(|x(t_0)|_\infty) + \sigma^d(d_\infty^{[t_0, t]})$ , we have to deal with the presence of  $\sigma^d(s)$  in the first term of  $\bar{\beta}$ . As in [8], we introduce the function

$$k(r, \sigma^d(s), t-t_0) = \min \{ \bar{\beta}(\sigma^x(r) + \sigma^d(s), t-t_0), \sigma^x(r) \}. \quad (19)$$

For any  $\alpha \in \mathcal{K}_\infty$ , the following inequality applies (see equations (98)-(100) in [19]):

$$k(r, \sigma^d(s), t-t_0) \leq \bar{\beta}(\sigma^x(r) + \alpha^{-1}(r), t-t_0) + \sigma^x \circ \alpha(\sigma^d(s)). \quad (20)$$

Then, functions  $\beta(r, s) = \bar{\beta}(\sigma^x(r) + \alpha^{-1}(r), s) \in \mathcal{KL}$  and  $\sigma(r) = \bar{\sigma}(r) + \sigma^x \circ \alpha(\sigma^d(r)) \in \mathcal{K}_\infty$  can be defined, thus obtaining for each  $i \in \mathcal{I}_N$  and for each  $t \geq t_0$

$$|x_i(t)| \leq \beta(|x(t_0)|_\infty, t-t_0) + \sigma(d_\infty^{[t_0, t]}). \quad (21)$$

Then, the sMS of (1) is proven. ■

Theorem 2 provides a stronger stability result with respect to Theorem 1, which is due to the stronger conditions on the stability of the subsystems. Moreover, it provides a similar results as [8, Theorem 1], extending it to the case where interconnection parameters  $\tilde{\gamma}_i > 1$  are present.

Theorem 1 and Theorem 2 ensure the existence of trajectory bounds for (1) that do not depend on the number of subsystems in the network. The obtained results highlight the importance of the knowledge of the local section of the interconnection graph surrounding an agent with non-ideal interconnection links. Indeed, for the LSS to be either sMMS or sMS, we need to investigate the capability of the subsystems to counterbalance the perturbations amplified by the non-ideal interconnections. As a consequence, dependence between the LSS stability and the trails describing how perturbations propagate through it exists. Also, this dependence results to affect the stability region of the LSS, which enlarges as the number of subsystems needed to dampen the non-ideal interconnection effects increases. Finally, we remark that this result is general, and it includes also the special case shown in [8] where  $\tilde{\gamma}_i < 1$  for each  $i \in \mathcal{I}_N$ .

#### IV. NUMERICAL EXAMPLE

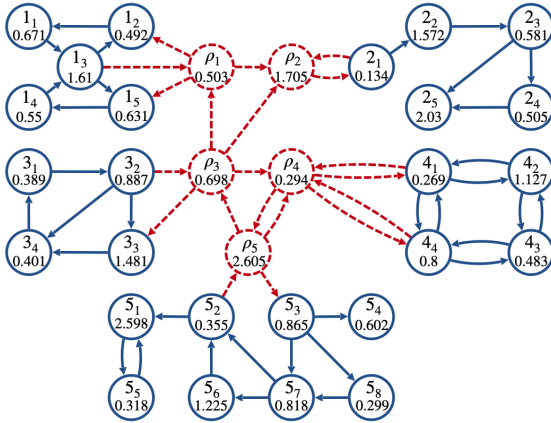


Fig. 2: The interconnection system considered as case study in Section IV

In order to validate the theoretical results shown in Section III, a numerical example is presented. We consider a modified version of the power network shown in [8, Section V], where an aggregate of microgrids composed by two different kinds of nodes is modeled. Fig. 2 illustrates the topology of the considered interconnection system, where first-level nodes (in blue, solid lines) represent agents interconnecting in a local area, while the second-level ones (in red, dashed lines) represent supervisors interacting with their own local area and with other supervisors. Differently from [8], here we consider the more constrained situation where not all the agents have a bidirectional interconnection. Let  $r \in \mathbb{N}$  be the total number of local areas in the network, and let  $i \in \{1, 2, \dots, r\}$ . The first-level nodes are denoted with  $i_j$ ,  $j \in \{1, 2, \dots, n_i\}$ , where  $n_i \in \mathbb{N}$  is the number of

agents in the  $i$ -th local area. The second-level nodes are denoted with  $\rho_i$ . The neighborhood of first and second-level nodes are denoted by  $\mathcal{N}_{i_j} \subseteq \{i_k | k = 1, 2, \dots, n_i\} \cup \{\rho_i\}$  and  $\mathcal{N}_{\rho_i} \subseteq \{i_j | j = 1, 2, \dots, n_i\} \cup \{\rho_k | k = 1, 2, \dots, N\}$ , respectively. Symbols  $x_{i_j}, x_{\rho_i} \in \mathbb{R}$  represent the state of the first and second-level nodes, respectively. In the same way,  $d_{i_j}, d_{\rho_i} \in [\underline{d}, \bar{d}] \subset \mathbb{R}$  represent the disturbances acting on the first and second-level nodes, respectively. The associated dynamics are

$$\dot{x}_{i_j} = -3x_{i_j} + d_{i_j} + \sum_{k \in K} a_k^{i_j} x_k + a_{\rho_i}^{i_j} x_{\rho_i} \quad (22a)$$

$$\dot{x}_{\rho_i} = -3x_{\rho_i} + d_{\rho_i} + \left( \prod_{l \in L} b_l^{\rho_i} |x_l| \right)^{\frac{1}{\|L\|}} + \left( \prod_{h \in H} c_h^{\rho_i} |x_h| \right)^{\frac{1}{\|H\|}} \quad (22b)$$

where  $K = \mathcal{N}_{i_j} \setminus \{\rho_i\}$ ,  $L = \mathcal{N}_{\rho_i} \cap \{i_j | j = 1, 2, \dots, n_i\}$ , and  $H = \mathcal{N}_{\rho_i} \cap \{\rho_p | p = 1, 2, \dots, r\}$ . We consider the coefficients  $a_k^{i_j}, a_{\rho_i}^{i_j} \in \mathbb{R}$ ,  $b_l^{\rho_i}, c_h^{\rho_i} \in \mathbb{R}^+$ ,  $\forall i_j, \rho_i$ .

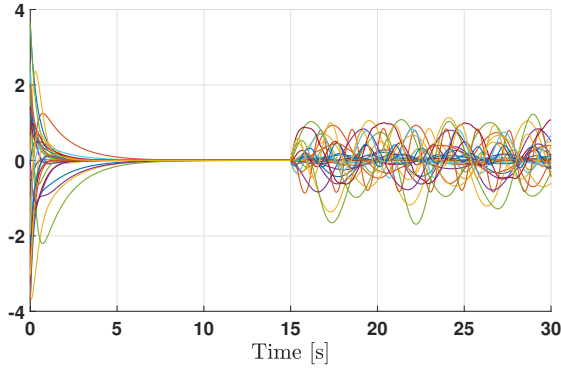
As shown in [8] (see equations (25)-(27)), by considering the quadratic Lyapunov functions  $V_{i_j} = x_{i_j}^2/2$  and  $V_{\rho_i} = x_{\rho_i}^2/2$  for (22a) and (22b), respectively, we can prove the existence of functions  $\beta_{i_j}, \beta_{\rho_i} \in \mathcal{KL}$ ,  $\sigma_{i_j}, \sigma_{\rho_i} \in \mathcal{K}_\infty$ , and  $\tilde{\gamma}_{i_j}, \tilde{\gamma}_{\rho_i} \in \mathbb{R}_{\geq 0}$  such that

$$|x_v(t)| \leq \beta_v(|x_v(t_0)|, t) + \sigma_v \left( |d_v(\cdot)|_\infty^{[t_0, t]} \right) + \tilde{\gamma}_v \max_{k \in \mathcal{N}_v} |x_k(\cdot)|_\infty^{[t_0, t]}, \quad (23)$$

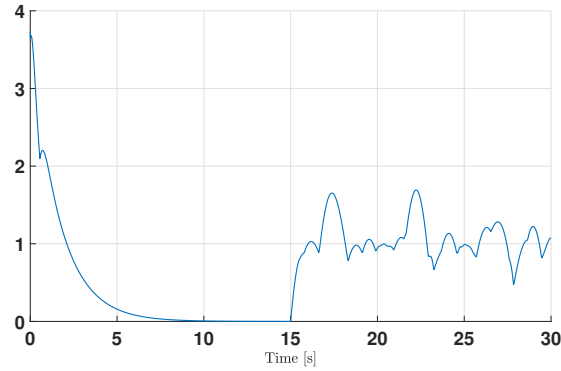
for each  $v \in \{i_j | i = 1, \dots, N, j = 1, \dots, n_i\} \cup \{\rho_i | i = 1, \dots, N\}$ . In particular, we get

$$\tilde{\gamma}_{i_j} = \sqrt{\frac{1}{5}} \sum_{k \in \mathcal{N}_{i_j}} |a_k^{i_j}|, \quad \tilde{\gamma}_{\rho_i} = \sqrt{\frac{1}{5}} (\Psi_b + \Psi_c), \quad (24)$$

where  $\Psi_b = \left( \prod_{l \in L} b_l^{\rho_i} |x_l| \right)^{\frac{1}{\|L\|}}$ ,  $\Psi_c = \left( \prod_{h \in H} c_h^{\rho_i} |x_h| \right)^{\frac{1}{\|H\|}}$ . Since for system (22) the ISS inequality (23) holds, the sMS of the interconnected power system can be proven. On the basis of Theorem 2, we need to analyse the interconnection parameters (24). In the example under analysis, the values of the interconnection parameters are shown within each node in Fig. 2. In particular, the subset  $\Gamma = \{1_3, 2_2, 2_5, 3_3, 4_2, 5_1, 5_6, \rho_2, \rho_5\}$  have the corresponding interconnection parameter between 1.127 and 2.605. Then, by investigating the local interconnection graph of each node in  $\Gamma$ , we derive the existence of a neighborhood able to counterbalance the effects of such agents, thus satisfying the condition in Assumption 2. Therefore, we can state the sMS of the considered interconnected system. Fig. 3b shows the numerical results obtained with respect to the interconnected system depicted in Fig. 2 in the form of maximum state perturbations. The simulation time is 30s and it is split into two phases: for  $0 \leq t < 15$ , the system starts with perturbed initial conditions, and no disturbances affecting the agents are present; for  $15 \leq t \leq 30$ , nodes  $1_3, 2_4, 3_2, 4_2, 5_5, 5_8, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5$  are affected by sinusoidal disturbances  $d_j(t) = A_j \sin(\omega_j t + \phi_j)$ , where



(a) Overall system trajectories.



(b) Maximum state perturbation:  $\max_{i \in \mathcal{I}_N} |x_i(t)|$ ,  $\forall t \in [0, 30]$ .

Fig. 3: Numerical simulations corresponding to the example in Fig. 2.

$A_j \in [1, 4]$ ,  $\omega_j \in [1, 4]$ , and  $\phi_j \in [0, \pi]$ . In Fig. 3b, we observe that the overall system is able to attenuate the perturbations due to both the non-ideal initial conditions and the sinusoidal disturbances. Therefore, the numerical results validate the theoretical ones, proving that the network is able to counterbalance the effects of the agents that locally amplify the perturbations.

## V. CONCLUSIONS

In this paper, we investigated the stability of a LSS with non-ideal interconnection links leading to the amplification of the perturbations propagating through it. We introduce the notion of scalable Marginal Mesh Stability (sMMS) requiring weaker conditions than the scalable Mesh Stability (sMS), and we provide sufficient conditions to be met by each subsystem to ensure the sMMS of the entire system. Moreover, we prove that if stronger conditions on the stability of each agent are verified, then the sMS of the same framework can be derived. The theoretical results show that there exists a dependence between the LSS stability and the trails describing how perturbations propagate through it. The theoretical framework derived here is used to analyze a network of microgrids with two kinds of nodes.

Future work will focus on the analysis of LSS where the number of agents varies over time (namely with plug-and-play subsystems), causing the rising of perturbations due to multiple transient phases in the presence of multiple agents

connecting/disconnecting.

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