

Logical Views of Bell Inequalities

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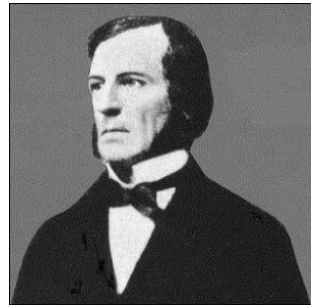


Outline

- George Boole's method in logic and probabilities
- Eigenlogic
- Different contexts: syntax and semantics
- Eigenlogic and the Born rule
- Bell inequalities and operators
- Logical boxes
- The Eigenlogic program

George Boole's method

George Boole



Boole's algebra of logic, in which algebraic techniques are applied to symbols representing classes, was successively systematized into the concept of a **Boolean algebra**.

As quoted from [c]

"If we look carefully at what Boole actually did in [a] [b] we find him carrying out operations of a different kind. Even though his starting algebraic equations were interpreted in logic, the allowed transformations often lead to equations without meaning in logic. Boole considered this acceptable so long as the end result could be given a meaning.

*What Boole did use was, if clarified, a **commutative ring with unit** and having **idempotents** which stood for classes. He used the ring operations and, in particular, its addition, which is not closed with respect to idempotency. Boole also freely used division to solve equations, introducing then a special method for extracting the logical content of the resulting quotient."*

He extended the logical method in the continuous interval $[0,1]$ to give one of the first mathematical formalizations of probabilities in [b].

[a] Boole, G. **The Mathematical Analysis of Logic. Being an Essay To a Calculus of Deductive Reasoning**, (1847)

[b] Boole, G. **An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities**, (1854)

[c] Hailperin, T. **Boole's Algebra isn't Boolean algebra. A description using modern algebra, of what Boole really did create.** Math. Mag. 54(4): 172–184 (1981).

George Boole : the “0” and the “1”

George Boole in 1847 [a] gave a mathematical symbolism for logical propositions.

The **conjunction** (AND) of 2 logical propositions X and Y is the product: $xy = yx$
 x (“elective” symbol) acts as a selection operator on y (also y on x)

applied on itself the proposition does not change resulting in: $x^2 = x$

this equation was considered by George Boole the “**fundamental law of thought**”! [b]

the only solutions of this equation are the numbers 0 and 1 representing “False” and “True” respectively.

Same equation written as: $x(1 - x) = 0$ the logical **law of non contradiction**

showing that x is an **idempotent** symbol (**projector**) orthogonal to $(1 - x)$ (its **complement**)

one also has $x + (1 - x) = 1$ the logical **law of the excluded middle**

[a] Boole, G. **The Mathematical Analysis of Logic. Being an Essay To a Calculus of Deductive Reasoning**, (1847)

[b] Boole, G. **An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities**, (1854)

using Boole's method for generating logical functions [*]

Boole's idempotent logical functions $f \in \{0,1\}$ are expressed in an **arithmetical** form (not modulo 2)

a logical function of two arguments is expressed by a **bilinear form** of the symbols x and y and the interpolation values $f(a, b)$ (truth values)

$$f(x, y) = f(0,0) (1 - x)(1 - y) + f(0,1) (1 - x)y + f(1,0) x(1 - y) + f(1,1) xy$$

Negation is the **complementation** by subtracting f from the number 1:

$$\bar{f} = 1 - f$$

generalizes to any number of arguments (arity)

TABLE 2. The sixteen two argument logical elective functions

TABLE 1. The four single argument logical elective functions

Function $f_i^{[1]}$	Operator	Truth table	Canonical form	Arithmetic form
$f_0^{[1]}$	F	0 0	0	0
$f_1^{[1]}$	\bar{A}	1 0	$(1 - x)$	$1 - x$
$f_2^{[1]}$	A	0 1	x	x
$f_3^{[1]}$	T	1 1	$(1 - x) + x$	1

Funct. $f_i^{[2]}$	Connective for A and B	Truth table	Canonical form	Arithmetic form
$f_0^{[2]}$	F	0 0 0 0	0	0
$f_1^{[2]}$	$NOR, \bar{A} \wedge \bar{B}$	1 0 0 0	$(1 - x)(1 - y)$	$1 - x - y + xy$
$f_2^{[2]}$	$\overline{A \leftarrow B}$	0 1 0 0	$(1 - x)y$	$y - xy$
$f_3^{[2]}$	\bar{A}	1 1 0 0	$(1 - x)(1 - y) + (1 - x)y$	$1 - x$
$f_4^{[2]}$	$\overline{A \Rightarrow B}$	0 0 1 0	$x(1 - y)$	$x - xy$
$f_5^{[2]}$	\bar{B}	1 0 1 0	$(1 - x)(1 - y) + xy$	$1 - y$
$f_6^{[2]}$	$XOR, A \oplus B$	0 1 1 0	$(1 - x)y + x(1 - y)$	$x + y - 2xy$
$f_7^{[2]}$	$NAND, \bar{A} \vee \bar{B}$	1 1 1 0	$(1 - x)(1 - y) + (1 - x)y + x(1 - y)$	$1 - xy$
$f_8^{[2]}$	$AND, A \wedge B$	0 0 0 1	xy	xy
$f_9^{[2]}$	$A \Leftrightarrow B$	1 0 0 1	$(1 - x)(1 - y) + xy$	$1 - x - y - 2xy$
$f_{10}^{[2]}$	B	0 1 0 1	$(1 - x)y + xy$	y
$f_{11}^{[2]}$	$A \Rightarrow B$	1 1 0 1	$(1 - x)(1 - y) + (1 - x)y$	$1 - x + xy$
$f_{12}^{[2]}$	A	0 0 1 1	$x(1 - y) + xy + xy$	x
$f_{13}^{[2]}$	$A \Leftarrow B$	1 0 1 1	$(1 - x)(1 - y) + x(1 - y) + xy$	$1 - y + xy$
$f_{14}^{[2]}$	$OR, A \vee B$	0 1 1 1	$(1 - x)y + x(1 - y) + xy$	$x + y + xy$
$f_{15}^{[2]}$	T	1 1 1 1	$(1 - x)(1 - y) + (1 - x)y + x(1 - y) + xy$	1

[*] Toffano, Z. **Eigenlogic in the Spirit of George Boole**. Logica Universalis, Birkhäuser-Springer, 14, 175–207 (2020).

George Boole : general logical inference method

From the proposition *men are rational animals* we want to find explicitly a definition of *rational beings* in terms of *men* and *animals*.

if x represent *men*, y *rational beings* and z *animals*, the proposition: *men are rational animals* will be represented by the equation:

$$x = yz$$

and y *rational beings* can be expressed by inversion :

$$y = \frac{x}{z}$$

Considering the interpolation method with $y = f(x, z)$ Boole interprets the result using $f(a, b) = \frac{a}{b}$:

$$y = f(x, z) = \frac{1}{1}zx + \frac{0}{1}z(1 - x) + \frac{1}{0}(1 - z)x + \frac{0}{0}(1 - z)(1 - x) = A + 0B + \frac{0}{0}C + \frac{1}{0}D$$

where the propositions are interpreted as:

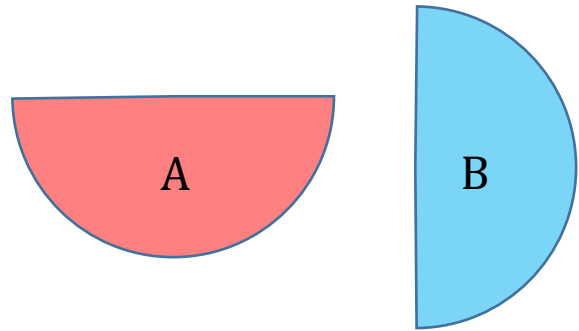
A	means	<i>all animals that are men</i>	
$0B$	means	<i>no animals that are not men</i>	
$\frac{0}{0}C$	means	<i>some none or all of beings that are neither animals nor men</i>	
$\frac{1}{0}D$	means	<i>men that are not animals do not exist</i>	(imposing the condition $D = (1 - z)x = 0$)

so in addition to the values 1 for “true” and 0 for “false”

Boole uses the peculiar $\frac{1}{0}$ to signify an “impossible” proposition and $\frac{0}{0}$ to signify an “indefinite” proposition.

logical forms and diagrams

Elementary propositions: A, B



SOP (Sum Of Products) canonical form

disjunction (\vee , OR) of conjunctions (\wedge , AND)

Conjunction, AND : $A \wedge B$
 in arithmetical form: ab

Disjunction, OR : $A \vee B = (\bar{A} \wedge B) \vee (A \wedge \bar{B}) \vee (A \wedge B)$
 in arithmetical form: $(1 - a)b + a(1 - b) + ab = a + b - ab$

Exclusive disjunction, XOR : $A \oplus B = (\bar{A} \wedge B) \vee (A \wedge \bar{B})$
 in arithmetical form: $(1 - a)b + a(1 - b) = a + b - 2ab$

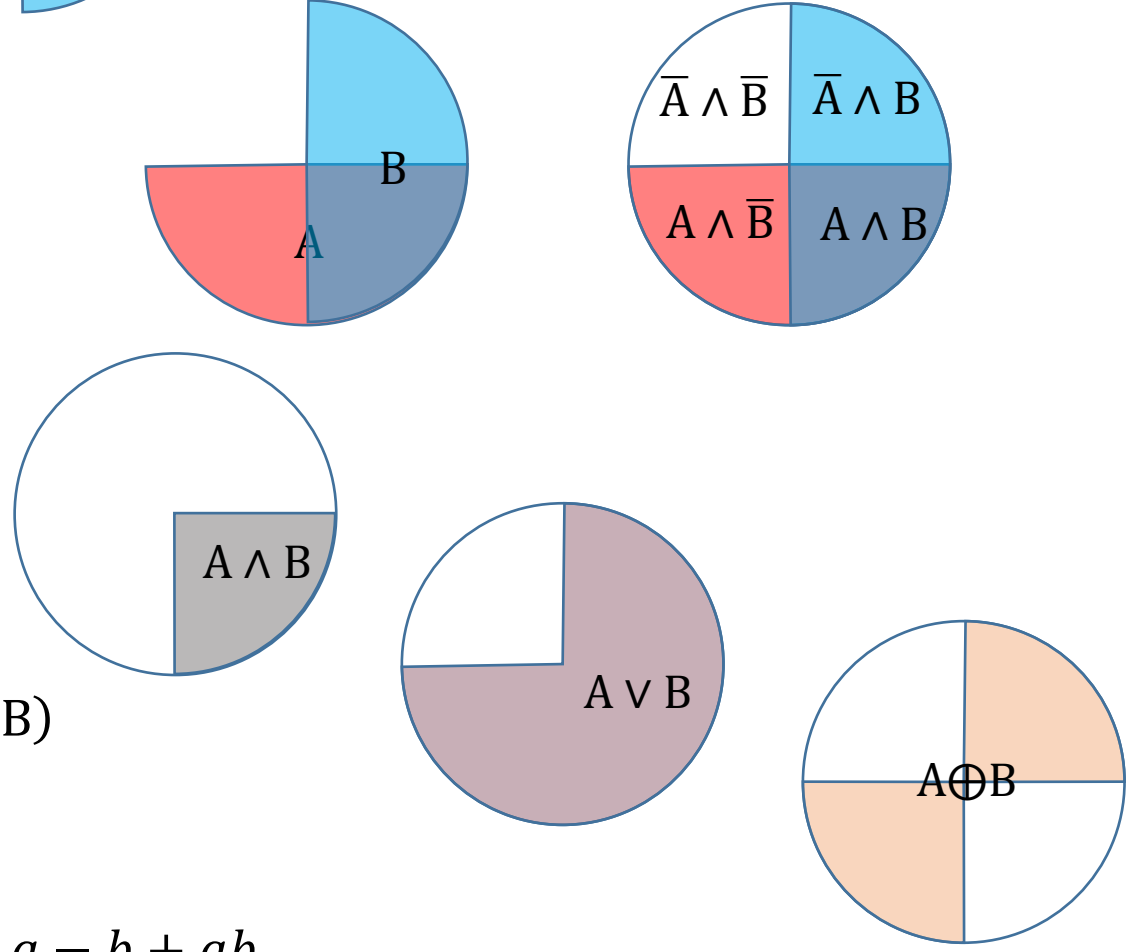
Reed-Muller canonical form

exclusive disjunction (\oplus , XOR) of conjunctions (\wedge , AND)

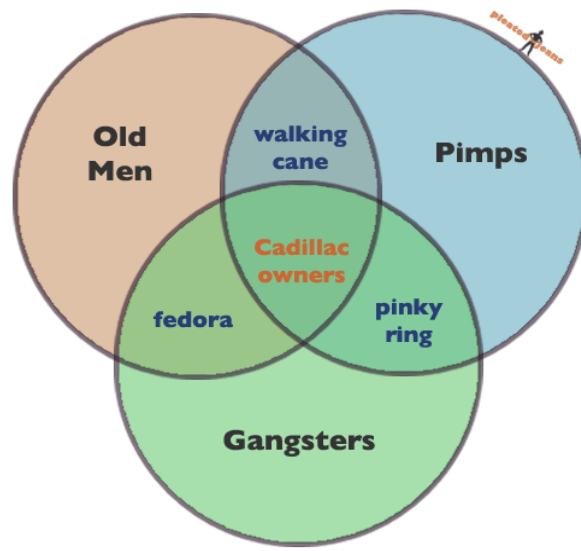
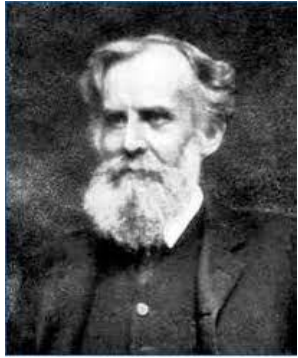
e.g. **Disjunction, OR :** $A \vee B = (A \oplus B) \vee (A \wedge B) = A \oplus B \oplus (A \wedge B)$

De Morgan logical duality

$\bar{A} \wedge \bar{B} = \overline{A \vee B}$
 in arithmetical form: $(1 - a)(1 - b) = 1 - (a + b - ab) = 1 - a - b + ab$



Venn Diagrams

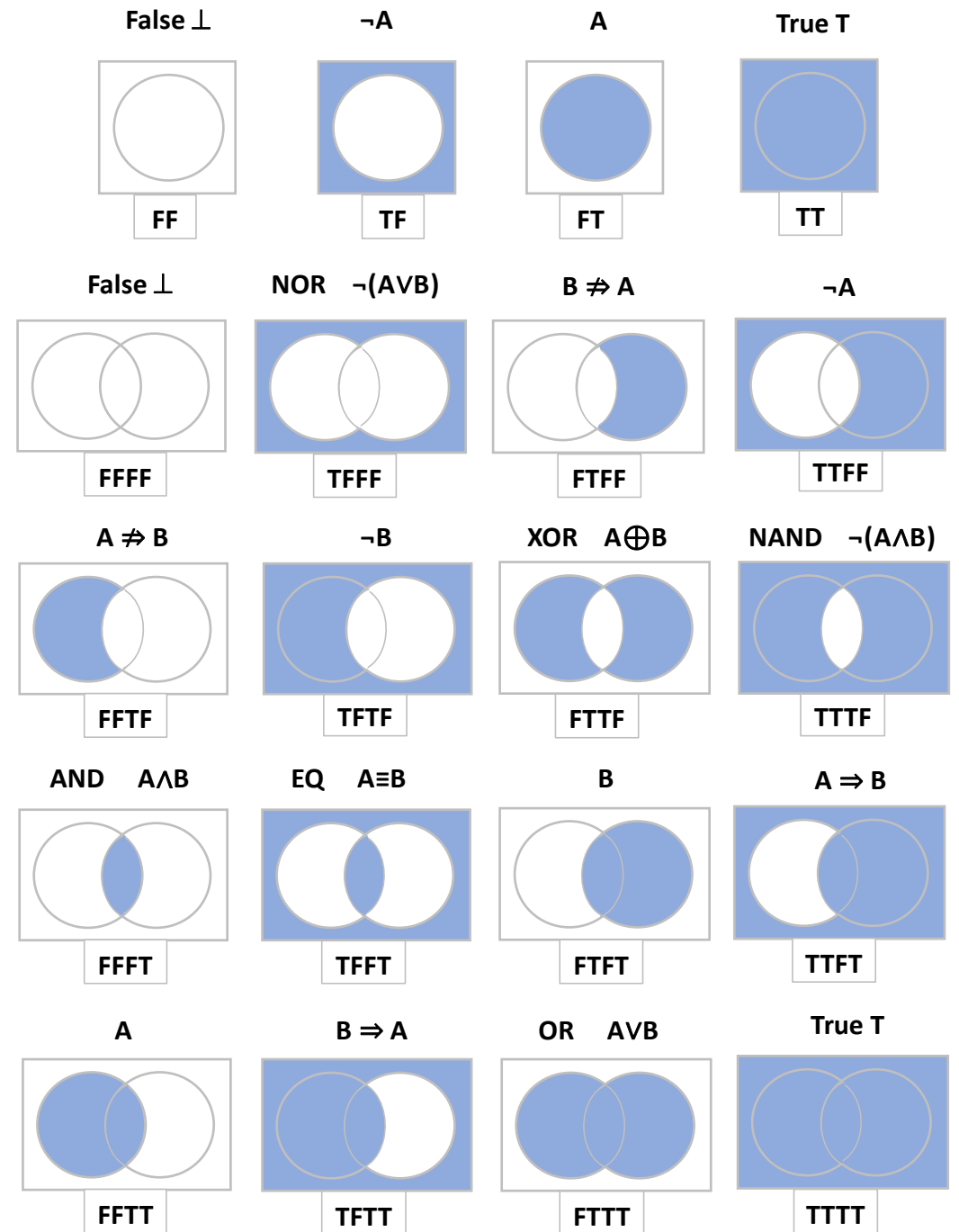


John Venn following exactly Boole's arithmetical approach illustrated all logical connectives in his **Venn diagrams** (1881) [*]

The diagrams have a direct correspondence with **set theory** by the operations of **Intersection** \cap and **Union** \cup of sets (here surfaces).

Are widely used in **probability theory** and **information theory** for the illustrations of different representations (independent, relative, conditional...)

[*] Venn, J. **Symbolic Logic**.
Macmillan and Company, London UK (1881)



George Boole : extension to probabilities

CHAPTER XVII.

DEMONSTRATION OF A GENERAL METHOD FOR THE SOLUTION OF PROBLEMS IN THE THEORY OF PROBABILITIES.

As can be inferred from the complete title of [*], Boole was also interested in the theory of probability and, in particular, the application of his logical system to the calculus of events.

It was the acknowledgement of the dual nature of the theory of probability, numerical and logical, that made [*] different from all previous treatises. He then introduced a new way of tackling it, which consists in:

“... substituting for events the propositions which assert that those events have occurred, or will occur; and viewing the element of numerical probability as having reference to the truth of those propositions...”

The general logic inference method is at the basis of his probabilistic reasoning for obtaining the “conditions of possible experience” through algebraic inequalities.

As an additional possibility in [*] Boole uses his peculiar expansions with $1/0$ to present an original approach to conditional probability.

[*] Boole, G. *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, (1854)

Boole's probability inequalities

The same method used in logic was extended by G. Boole in the continuous interval $[0,1]$.

The well known property :

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

transforms as in propositional logic for disjunction \vee (OR), and conjunction \wedge (AND).

If $P(A) = p$ and $P(B) = q$ with $p, q \in [0,1]$ one has :

$$P(A \vee B) = p + q - pq \leq p + q = P(A) + P(B)$$

also

$$P(A \vee B \vee C) = p + q + r - pq - pr - qr + pqr \leq p + q + r = P(A) + P(B) + P(C)$$

increasing the number of propositions with \vee (OR) one gets the *inclusion-exclusion principle* outlined by **Henri Poincaré** [*].

From the above considerations one can thus express the **Boole inequality**:

$$P\left(\bigvee_{i=1}^N A_i\right) \leq \sum_{i=1}^N P(A_i)$$

By similar reasoning one obtains also the **Bonferroni's inequality**:

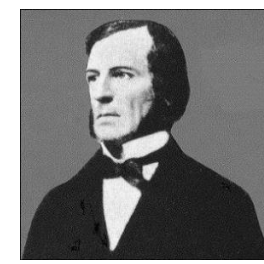
$$P\left(\bigvee_{i=1}^N A_i\right) \geq \sum_{i=1}^N P(A_i) - \sum_{i < j} P(A_i \wedge A_j)$$

[*] Poincaré, H. **Calcul des Probabilités**; Gauthier-Villars: Paris, France, 1912

Eigenlogic

operators in Logic

In 1847 **G. Boole** uses symbols (elective) that act as idempotent operators
less known: in 1848 in [a], he gave a logical interpretation of unitary **quaternions**.



C. S. Peirce used matrices and quaternion algebra to build his logical formalism at the end of the XIXth century.

In 1921 **L. Wittgenstein** states in the Tractatus [b] that all propositions can be derived by repeated application of the operator **N** to the elementary propositions.

In 1924, **M. Schönfinkel** [c] introduced an operator based method in logic.

H. Curry named it successively **Combinatory Logic** and improved the method in [d] which lead successively to **lambda calculus** introduced **A. Church**.



In 1943 **E.L. Post** [e] created a string-manipulation system (**Post production system**) and proved the remarkable **Normal-form Theorem** $g\$ \rightarrow \h



[a] Boole, G. **Notes on quaternions**. Philos. Mag, 33, 278–280, (1848)

[b] Wittgenstein, L. **Tractatus Logico-Philosophicus**, prop. 6.001, Routledge (1921)

[c] Schönfinkel, M. **Über die Bausteine der mathematischen Logik**. Math. Ann, 92, 305–316. (1924)

[d] Curry, H.B.; Feys, R. **Combinatory Logic**; North-Holland Co: Amsterdam, The Netherlands, (1958)

[e] Post, E. **Formal Reductions of the General Combinatorial Decision Problem**, American Journal of Mathematics 65 (2), 197-215. (1943)

quantum logic : projections as propositions

M. H. Stone gave the conditions for operations on projectors and commutativity [a] and established that each binary logical proposition corresponds by duality to the set of all its true valuations (**Stone Duality**).

J. von Neumann considered measurement projection operators as propositions in 1932 [b] and also stated that a quantum state $|\psi\rangle$ can be represented by a **density matrix** (rank-1 projection operator) :

$$\rho = |\psi\rangle\langle\psi|$$

Quantum Logic proposed by **G. Birkhoff** and **J. von Neumann** in 1936 [c] suggested the replacement of Boolean algebras with the lattice of closed subspaces of a (finite) Hilbert space.

based on projection measurements it is a **non-Boolean** logic and fails to meet distributive properties as expected by quantum mechanics.
problem: no satisfying way to formulate logical implication



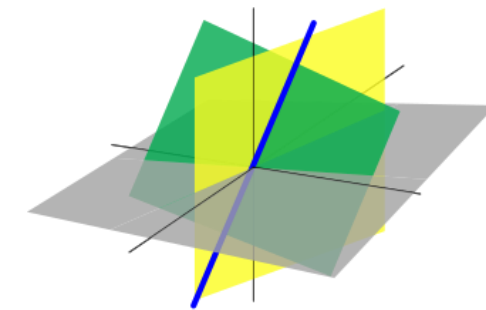
M.H. Stone



G. Birkhoff



J. Von Neumann



[a] Marshall Harvey Stone: **Linear Transformations in Hilbert Space and Their Applications to Analysis**, p.70: “Projections”, (1932)

[b] John von Neumann. **Mathematical Foundations of Quantum Mechanics**, Eng. Transl. (1955), p.249: “Projectors as Propositions”, (1932)

[c] Garret Birkhoff, John von Neumann, **The Logic of Quantum Mechanics**. The Annals of Mathematics, 2nd Ser., 37 (4), 823-843 (1936)

Eigenlogic

Eigenlogic: a logical method using operators in linear algebra [a,b,c]

logical operators \Leftrightarrow **logical connectives**

eigenvalues of logical operators \Leftrightarrow **truth values**

eigenvectors of logical operators \Leftrightarrow **interpretations (propositional cases)**

Eigenlogic uses the Kronecker product to scale-up to more logical arguments (arity).

A single seed operator generates the entire logic.

[a] Dubois, F., Toffano, Z., **Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems**,
in: de Barros J., Coecke B., Pothos E. (eds) Quantum Interaction. QI 2016. Lecture Notes in Computer Science, vol 10106. Springer. (2017)

[b] Toffano, Z., **Eigenlogic in the Spirit of George Boole**, Logica Universalis, Birkhäuser-Springer, 14, 175–207. (2020)

[c] Toffano Z, Dubois F., **Adapting Logic to Physics: The Quantum-Like Eigenlogic Program**, Entropy. ; 22(2):139. (2020)

Cayley-Hamilton theorem and matrix interpolation

The Eigenlogic seed operator Λ can be any operator with m non-degenerate eigenvalues λ_i ,

using **Lagrange matrix interpolation** the projector of each eigenstate is given by:

$$|\lambda_i\rangle\langle\lambda_i| = \Pi_{\lambda_i}(\Lambda) = \prod_{j=1, j \neq i}^m \frac{\Lambda - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}$$

is a polynomial in Λ up to the power $m - 1$ and is represented by a $m \times m$ square matrix.

The **Cayley-Hamilton theorem** says that any finite matrix is the solution of its own characteristic equation showing that the above development is unique.

A logical operator for arity-1 is then given by the spectral decomposition with truth-values $f(\lambda_j) \in \{\dots \lambda_i \dots\}$:

$$\mathbf{F}_L = \sum_{j=1}^m f(\lambda_j) \Pi_{\lambda_j}(\Lambda)$$

Scaling to higher arity is obtained by extending by the Kronecker product the seed operator Λ with the identity.

Eigenlogic: one-qubit Boolean logical operators

The qubits $|1\rangle$ and $|0\rangle$ define the **computational basis** (the “z” base): $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
eigenvectors of the **Pauli matrix** $\sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \text{diag}(+1, -1)$

Choice of the logical seed projector $\Pi = |1\rangle\langle 1|$ (density matrix of qubit $|1\rangle$)

Logical operators as a linear development (**equivalent to Boole’s method**) gives the **spectral decomposition** :

$$F = f(0)(\mathbb{I} - \Pi) + f(1)\Pi = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} = \text{diag}(f(0), f(1))$$

the cofactors $f(0)$ and $f(1)$ are the eigenvalues *i.e.* the truth values of the logical connective.

Negation is obtained by complementation (subtracting from the identity operator): $\bar{F} = \mathbb{I} - F$

other choices of logical bases are possible: e.g. the “x” base with the seed $\Pi_- = |-\rangle\langle -|$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Eigenlogic: two-qubit Boolean logical operators

Making use of the **Kronecker product** \otimes to scale up to more arguments (as done in quantum computing)

Scaling to 2-qubit logical operators with the 4 basis projection operators (pure quantum-state density matrices):

$$\begin{aligned}\rho_{11} &= |11\rangle\langle 11| = \Pi \otimes \Pi & ; & & \rho_{10} &= |10\rangle\langle 10| = \Pi \otimes (\mathbb{I} - \Pi) ; \\ \rho_{01} &= |01\rangle\langle 01| = (\mathbb{I} - \Pi) \otimes \Pi & ; & & \rho_{00} &= |00\rangle\langle 00| = (\mathbb{I} - \Pi) \otimes (\mathbb{I} - \Pi)\end{aligned}$$

All 16 logical arity-2 operators are directly obtained by the bilinear development (**G. Boole's method**)

Giving the **spectral decomposition of the operator** :

$$\begin{aligned}\mathbf{F} &= f(0,0)|00\rangle\langle 00| + f(0,1)|01\rangle\langle 01| + f(1,0)|10\rangle\langle 10| + f(1,1)|11\rangle\langle 11| = \\ &= f(0,0)(\mathbb{I} - \Pi) \otimes (\mathbb{I} - \Pi) + f(0,1)(\mathbb{I} - \Pi) \otimes \Pi + f(1,0)\Pi \otimes (\mathbb{I} - \Pi) + f(1,1)\Pi \otimes \Pi \\ &= \text{diag}(f(0,0), f(0,1), f(1,0), f(1,1))\end{aligned}$$

the truth values are $f(x, y) \in \{0,1\}$

Eigenlogic elementary propositions and logical connectives

In **propositional logic** one defines the **atomic propositions** A and B in a **well-formed-formula**. From the elementary propositions A and B all other compound propositions can be derived.

$$\mathbf{A} = \mathbf{\Pi} \otimes \mathbf{I} = \text{diag}(0,0,1,1) \quad , \quad \mathbf{B} = \mathbf{I} \otimes \mathbf{\Pi} = \text{diag}(0,1,0,1)$$

atomic propositions In Eigenlogic correspond to the extensions of the seed operator $\mathbf{\Pi}$ with the identity operator \mathbf{I} , **a major difference with “traditional quantum logic” where atomic propositions correspond to quantum states $|\psi\rangle$ (the corresponding density matrices $|\psi\rangle\langle\psi|$ are rank-1 projectors).**

directly from \mathbf{A} and \mathbf{B} all other compound logical operators are derived:

Conjunction (AND, \wedge) $\mathbf{F}_{\text{AND}} = \mathbf{F}_{\mathbf{A} \wedge \mathbf{B}} = \mathbf{A} \cdot \mathbf{B} = \mathbf{\Pi} \otimes \mathbf{\Pi} = \text{diag}(0,0,0,1)$

Disjunction (OR, \vee) $\mathbf{F}_{\text{OR}} = \mathbf{F}_{\mathbf{A} \vee \mathbf{B}} = \mathbf{A} + \mathbf{B} - \mathbf{A} \cdot \mathbf{B} = \text{diag}(0,1,1,1)$

Negation is simply obtained by subtracting from the identity operator \mathbf{I} :

$$\mathbf{F}_{\text{NAND}} = \mathbf{I} - \mathbf{F}_{\text{AND}} = \text{diag}(1,1,1,0) \quad ; \quad \text{Equivalence} \quad \mathbf{F}_{\Leftrightarrow} = \mathbf{I} - \mathbf{F}_{\text{XOR}} = \text{diag}(1,0,0,1)$$

Similar approach used for QECC (Quantum Error Correcting Codes) in **[*]**

[*] Aggarwal, V., Calderbank, R. **Boolean Functions, Projection Operators, and Quantum Error Correcting Codes**. In: IEEE Proceedings ISIT 2007 (International Symposium Information Theory), Nice, France, pp. 2091–2095 (2007)

Eigenlogic and the Born rule

when the logical input is not an eigenstate

In Eigenlogic, the mean value of a logical operator will provide the truth value of the associated logical proposition. For the logic projection operator F measured on a quantum state $|\psi\rangle$, the mean value leads directly to a probability measure by the Born rule:

$$P_{|\psi\rangle} = \langle\psi|F|\psi\rangle = \text{Tr}(\rho \cdot F) \quad \text{with } \rho = |\psi\rangle\langle\psi| \text{ the density matrix.}$$

e.g. general quantum state is expressed by a linear combination over the 2-qubit computational basis:

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

when more than one coefficient is non-zero one is in a superposition of logical inputs.

The *interpretable* case, corresponding to sharp truth values, is obtained for input states belonging to the logical eigenspace of the Eigenlogic operators.

$$\text{For example for the eigenstate } |\psi\rangle = |01\rangle : \quad \langle 01|F_{\text{AND}}|01\rangle = 0, \quad \langle 01|F_{\text{OR}}|01\rangle = 1, \quad \langle 01|F_{\text{XOR}}|01\rangle = 1$$

$$\text{but for } |\varphi\rangle = \frac{1}{\sqrt{3}}(|01\rangle - |10\rangle + |11\rangle) : \quad \langle\varphi|F_{\text{AND}}|\varphi\rangle = \frac{1}{3}, \quad \langle\varphi|F_{\text{OR}}|\varphi\rangle = 1, \quad \langle 01|F_{\text{XOR}}|01\rangle = \frac{2}{3}$$

conjunction, disjunction and material implication

Bloch sphere in Hilbert space

A generic qubit state on the Bloch sphere: $|\phi\rangle = \sin\frac{\theta}{2}|0\rangle + e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$,

The quantum average (Born rule) of the logical projector is: $p(A) = \langle\phi|\Pi|\phi\rangle = \cos^2\frac{\theta}{2}$

The Born rule for **A** and **B** by performing the quantum average on the compound state :

*

$$|\psi\rangle = |\phi_p\rangle \otimes |\phi_q\rangle \quad \text{with} \quad p = \left(\cos\frac{\theta_p}{2}\right)^2 \quad \text{and} \quad q = \left(\cos\frac{\theta_q}{2}\right)^2$$

we get $P(A) = \langle\psi|\mathbf{A}|\psi\rangle = p(1 - q) + p \cdot q = p$; $P(B) = \langle\psi|\mathbf{B}|\psi\rangle = q$

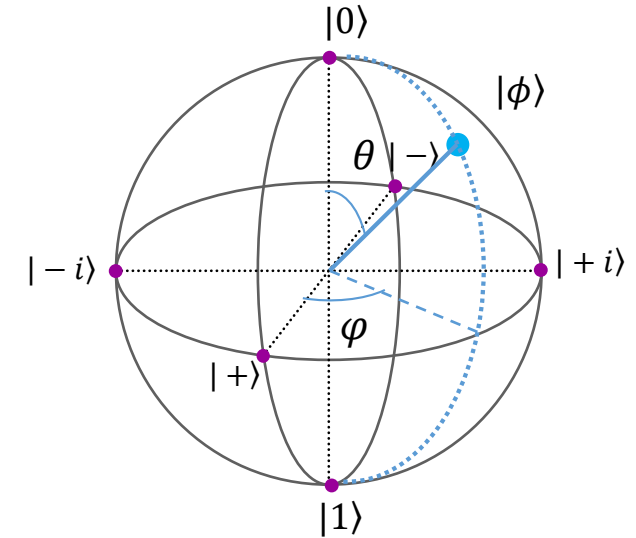
Conjunction (AND) $P(A \wedge B) = \langle\psi|\mathbf{A} \cdot \mathbf{B}|\psi\rangle = \langle\psi|\Pi \otimes \Pi|\psi\rangle = p \cdot q = P(A) \cdot P(B)$

Disjunction (OR) $P(A \vee B) = p + q - p \cdot q = P(A) + P(B) - P(A) \cdot P(B)$

Material Implication $P(A \Rightarrow B) = 1 - p + p \cdot q = 1 - P(P) + P(P) \cdot P(Q)$

The disjunction $P(A \vee B)$ corresponds to the **inclusion-exclusion** expression for probabilities

In the language of fuzzy logic these are **Product t-norms** (triangular norm as a product for the fuzzy conjunction)



Different contexts: syntax and semantics

changing the paradigm: using values $\{+1, -1\}$ instead of $\{0, 1\}$

The **polar alphabet** $\{+1, -1\}$ has the following correspondence with the Booleans $\{0, 1\}$:

$$+1 \text{ (spin up)} \leftrightarrow 0 \text{ : "False"} \quad ; \quad -1 \text{ (spin down)} \leftrightarrow 1 \text{ : "True"}$$

this **binary reversible logic alphabet** is often used (implicitly) in **Ising spin models** and **neural networks**.

considering $x \in \{0,1\}$ one has $u \in \{+1, -1\}$ if and only if $u = 1 - 2x = (-1)^x = e^{i\pi x}$

for **operators** the equivalent form is given by an isomorphism the **Householder Transform** :

$$\mathbf{G} = \mathbb{I} - 2\mathbf{F} = (-1)^{\mathbf{F}} = e^{i\pi\mathbf{F}} = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{2}\mathbf{G}}$$

projection operators \mathbf{F} (eigenvalues $\{0,1\}$) \Leftrightarrow reversible **involution operators** \mathbf{G} (eigenvalues $\{+1, -1\}$)

\mathbf{G} and \mathbf{F} have the same eigenvectors

Comparable approach using the notion of Fourier analysis of Boolean functions in [*]

[*] Montanaro, A., Osborne, T.J., **Quantum Boolean Functions**, Chicago Journal of Theoretical Computer Science, 1, pp. 1–45 (2010)

binary operator truth tables

truth tables show

valuations: \models in **semantics**

logical connectives are used for

deductions: \vdash in **syntax**

consistency (if $\vdash A$ then $\models A$)

and

completeness (if $\models A$ then $\vdash A$)

are equivalent for the propositional calculus :

completeness theorem

Is not valid for 1st order logic as shown by **Gödel's incompleteness theorem**

logical connective for P, Q	truth table {F, T}: {0, 1} or {+1, -1}	{0, 1} projection logical operator	{+1, -1} involution logical operator
False F	F F F F	0	+ I
NOR	F F F T	$I - P - Q + PQ$	$(1/2) (+I - U - V - UV)$
$P \neq Q$	F F T F	$Q - PQ$	$(1/2) (+I - U + V + UV)$
$\neg P$	F F T T	$I - P$	- U
$P \neq Q$	F T F F	$P - PQ$	$(1/2) (+I + U - V + UV)$
$\neg Q$	F T F T	$I - Q$	- V
XOR ; $P \oplus Q$	F T T F	$P + Q - 2 PQ$	$UV = Z \otimes Z$
NAND ; $P \uparrow Q$	F T T T	$I - PQ$	$(1/2) (-I - U - V + UV)$
AND ; $P \wedge Q$	T F F F	$PQ = \Pi \otimes \Pi$	$(1/2) (+I + U + V - UV)$
$P \leftrightarrow Q$	T F F T	$I - P - Q + 2 PQ$	- UV
Q	T F T F	$Q = I \otimes \Pi$	$V = I \otimes Z$
$P \Rightarrow Q$	T F T T	$I - P + PQ$	$(1/2) (-I - U + V - UV)$
P	T T F F	$P = \Pi \otimes I$	$U = Z \otimes I$
$P \Leftarrow Q$	T T F T	$I - Q + PQ$	$(1/2) (-I + U - V - UV)$
OR ; $P \vee Q$	T T T F	$P + Q - PQ$	$(1/2) (-I + U + V + UV)$
True T	T T T T	I	- I

Boole's logical interpretation of unitary quaternions

In a short note [\[*\]](#), just one year after his introduction of mathematical logic Boole states:

*“Signs employed as instruments of reasoning may be considered as the representatives of operations...upon examination it will be found that these systems of interpretation are founded upon a principle of naming, as the one which I have proposed is founded upon a principle of operation.
... the symbolical forms of common language as exhibited in the calculus of logic may indifferently be referred to the one or the other of these modes of conception.”*

So he discusses implicitly the duality between naming with a sign which represents logical semantics and operation which represents logical syntax.

George Boole illustrates this logical interpretation using a unitary quaternion defined by:

$$\mathbf{q} = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \quad \text{with} \quad w^2 + x^2 + y^2 + z^2 = 1$$

$$\mathbf{i}, \mathbf{j}, \mathbf{k} \text{ is the quaternion basis verifying: } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} = -1, \mathbf{i} \cdot \mathbf{j} = \mathbf{k}, \mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i}$$

Which shows the non commutativity of the quaternion basis and the role of the sign $-$.

remark: the quaternion basis is isomorphic to the quantum Pauli matrices. $\mathbf{i} = -i\hat{\sigma}_x, \mathbf{j} = -i\hat{\sigma}_y, \mathbf{k} = -i\hat{\sigma}_z$

[\[*\]](#) Boole, G. *Notes on quaternions*. Philos. Mag., 33, 278–280 (1848)

Eigenlogic syntax-semantic duality and non-commutativity

Considering the 2 eigenstates $|\pm z\rangle$ of the Pauli operator σ_z with eigenvalues ± 1

and using the **anti-commutativity** of the Pauli operators:

$$\sigma_x \cdot \sigma_z = -\sigma_z \cdot \sigma_x$$

$$\sigma_x \cdot \sigma_z |\pm z\rangle = (\pm 1)\sigma_x |\pm z\rangle = -\sigma_z \cdot \sigma_x |\pm z\rangle \quad \text{gives} \quad \sigma_z(\sigma_x |\pm z\rangle) = (\mp 1)(\sigma_x |\pm z\rangle)$$

so $(\sigma_x |\pm z\rangle)$ are eigenstates of σ_z with eigenvalues ∓ 1 and correspond to the eigenstates $|\mp z\rangle$

Identifying $|+z\rangle$ with qubit $|0\rangle$ and $|-z\rangle$ with qubit $|1\rangle$ gives : $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$

this operation corresponds to **logical binary negation**.

So for these operators the basic logical operation of binary negation is a consequence of anti-commutativity

In this very simple example using the Pauli matrices σ_x and σ_z as Eigenlogic operators, one has simultaneously:

- a **semantic representation** by the eigenstructure (eigenvalues ± 1 and eigenvectors $|\pm z\rangle$) of Pauli matrix σ_z
- a **syntactic representation** by a permutation operation represented by the action of Pauli matrix σ_x on $|\pm z\rangle$.

the Pauli matrices σ_x and σ_z represent MUB (mutually unbiased) basis contexts. The transformation from one to the other is the **quantum Fourier transform** (here in 2 dimensions the Hadamard gate).

syntax and semantics for many-valued operators (qudits)

The logical complementation can be generalized for a d -dimensional multi-level system (**qudit**)

using the **generalized Pauli operators** given by the **Weyl-Heisenberg pairs** X_d and Z_d

$$\mathbf{Z}_d |j\rangle = \omega_d^j |j\rangle \quad \text{with} \quad \omega_d = e^{i\frac{2\pi}{d}} \quad ; \quad \mathbf{X}_d |j\rangle = |j+1\rangle \quad \text{with} \quad \mathbf{Z}_d^d = \mathbf{X}_d^d = \mathbb{I}_d$$

X_d and Z_d possess the same eigenvalues and verify : $\mathbf{Z}_d \cdot \mathbf{X}_d = \omega_d \mathbf{X}_d \cdot \mathbf{Z}_d$

the action of the **shift operator** X_d on the state $|j\rangle$, which is an eigenstate of Z_d , gives the state $|j+1\rangle$
so by applying successively this operator one can generate all the other states of the basis.

The semantics is here represented by the eigenstructure of Z_d , the eigenvalues ω_d are the d^{th} roots of unity

The syntax is represented by X_d corresponding to a **many-valued negation** as used by **E.L. Post** in 1921.

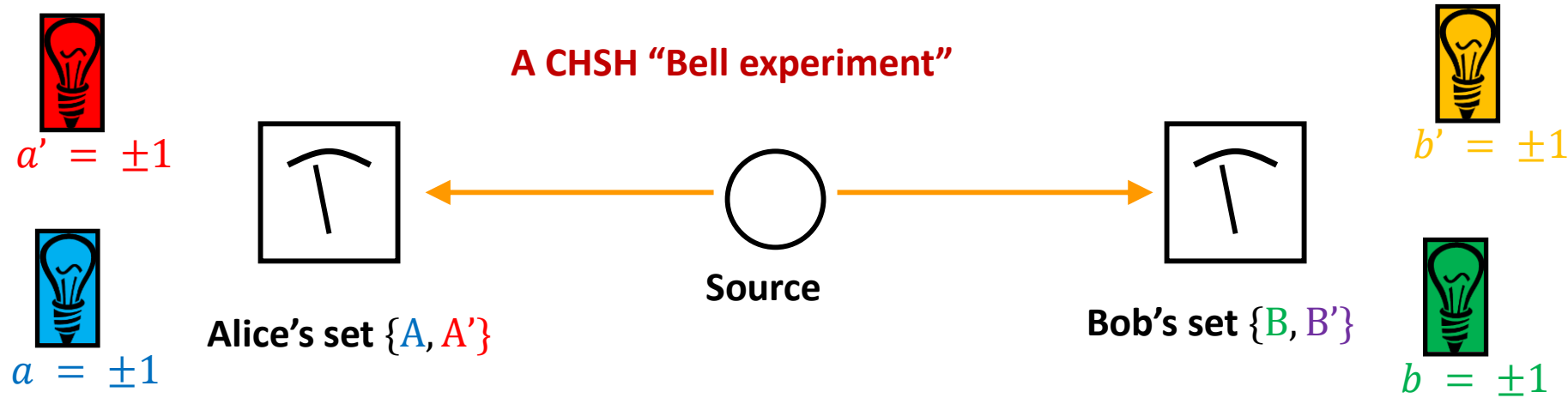
Passing from Z_d to X_d is the **Discrete Fourier Transform** operator QFT_d (**Quantum Fourier Transform**)

$$(QFT_d)_{ij} = \frac{1}{\sqrt{d}} \omega_d^{ij} \quad ; \quad QFT_d^4 = \mathbb{I}_d \quad ; \quad QFT_d^{-1} \cdot \mathbf{Z}_d \cdot QFT_d = \mathbf{X}_d$$

The Quantum Fourier Transform can be seen as a mediator between logical syntax and logical semantics.

Bell inequalities and Operators

correlations of 2 quantum particles: CHSH Bell-inequality [*]



Local properties ($a = \pm 1, a' = \pm 1, b = \pm 1, b' = \pm 1$)

in all 16 cases (deterministic) the expression:

$$ab + ab' + a'b - a'b' = a(b + b') + a'(b - b') = \pm 2$$

CHSH-Bell inequality requires 16 measurements giving (average on a great number of measurements):

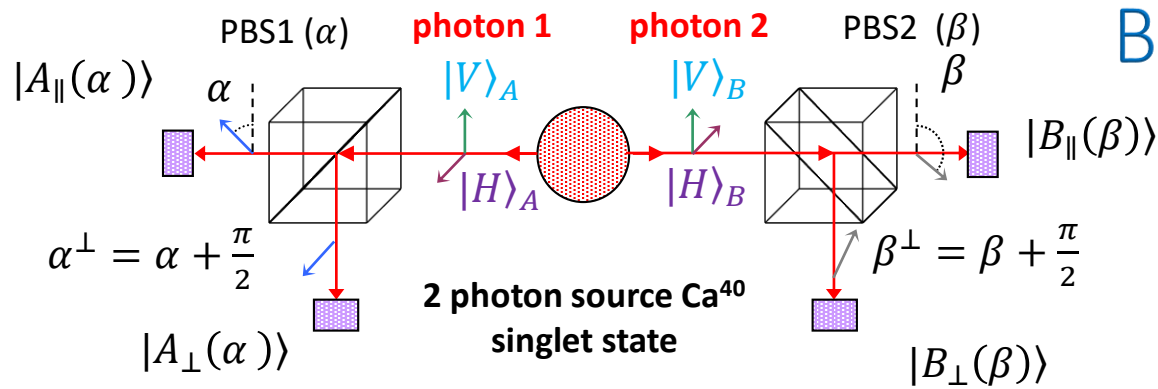
$$|S| = | \langle ab + ab' + a'b - a'b' \rangle | \leq 2$$

But Quantum Mechanics allows: $2 < S \leq 2\sqrt{2} = 2.83$

so violates the CHSH Inequality > 2 !

[*] J.F. Clauser; M.A. Horne; A. Shimony; R.A. Holt, **Proposed experiment to test local hidden-variable theories**, Phys. Rev. Lett., 23 (15): 880–4, (1969)

Bell inequality: the Orsay experiment [*]



[*] A. Aspect; P. Grangier; G. Roger, **Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities**. Phys. Rev. Lett. 49 (2): 91-4, (1982)

$C(\alpha, \beta)$: coincidence photon counting rate function of the polarization angles α and β

Then the 2-photon correlation is defined:

$$E(\alpha, \beta) = \frac{C(\alpha, \beta) + C(\alpha^\perp, \beta^\perp) - C(\alpha^\perp, \beta) - C(\alpha, \beta^\perp)}{C(\alpha, \beta) + C(\alpha^\perp, \beta^\perp) + C(\alpha^\perp, \beta) + C(\alpha, \beta^\perp)}$$

The **Bell CHSH inequality** is a function of the correlations E for 4 experimental settings $\{\alpha_i, \beta_j\}, i, j \in \{1, 2\}$:

the **Bell parameter**: $|S| = |E(\alpha_1, \beta_1) + E(\alpha_1, \beta_2) + E(\alpha_2, \beta_1) - E(\alpha_2, \beta_2)|$

Classically the Bell inequality verifies: $-2 \leq |S| \leq +2$

for the polarization angles : $\beta_1 - \alpha_1 = \frac{\pi}{8}, \alpha_2 - \beta_1 = \frac{\pi}{8}, \beta_2 - \alpha_2 = \frac{\pi}{8}$

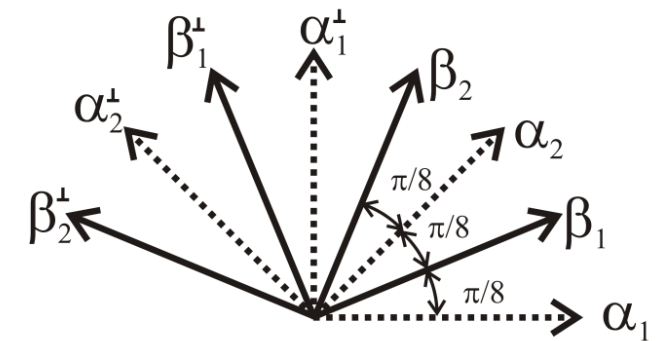
we have a violation of the Bell CHSH inequality

$$|S| \geq 2$$

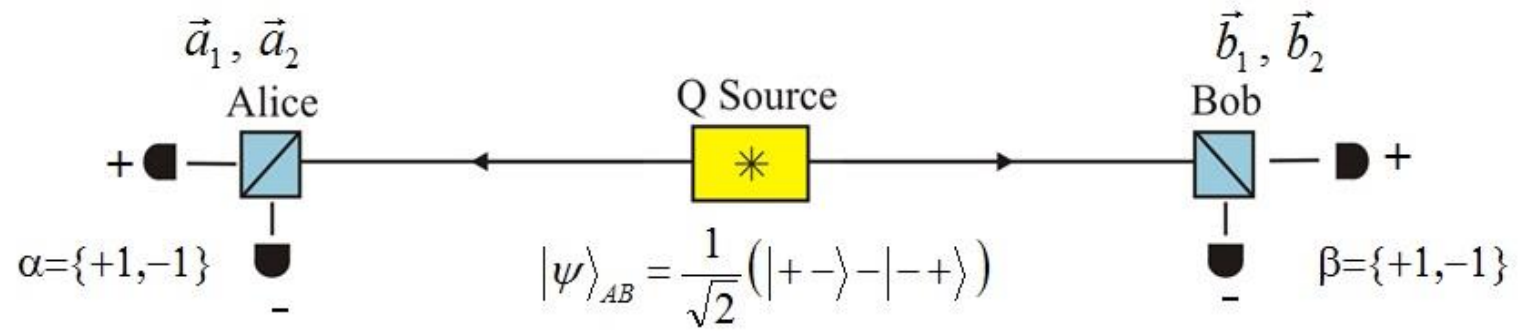
with the maximal value
obtained with an **entangled state (Bell state)**

$$|S|_{max} = 2\sqrt{2}$$

$$|\Psi^-\rangle_s = \frac{1}{\sqrt{2}} (|H\rangle_1 |V\rangle_2 - |V\rangle_1 |H\rangle_2)$$



The maximal CHSH quantum inequality violation



Let's consider the case of the following measuring observables function of the Pauli matrices σ_x and σ_z :

$$\mathbf{a}_1 = \sigma_z ; \quad \mathbf{a}_2 = \sigma_x ; \quad \mathbf{b}_1 = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) ; \quad \mathbf{b}_2 = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x)$$

the set of 2 observables on both sides are MUB (maximally unbiased) systems corresponding to maximally incompatible contexts.

The Bell parameter S as the average of the observable \mathcal{S} by grouping the 4 terms gives :

$$\mathcal{S} = \mathbf{a}_1 \otimes \mathbf{b}_1 + \mathbf{a}_1 \otimes \mathbf{b}_2 + \mathbf{a}_2 \otimes \mathbf{b}_1 - \mathbf{a}_2 \otimes \mathbf{b}_2 = \sqrt{2}(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x) = \sqrt{2} \begin{pmatrix} +1 & 0 & 0 & +1 \\ 0 & -1 & +1 & 0 \\ 0 & +1 & -1 & 0 \\ +1 & 0 & 0 & +1 \end{pmatrix}$$

The eigenvalues of \mathcal{S} are $\{+2\sqrt{2}, 0, 0, -2\sqrt{2}\}$ so for 2 eigenvectors we have : $S = |\langle \Psi | \mathcal{S} | \Psi \rangle| = 2\sqrt{2}$

violates the CHSH inequality. These eigenvectors are the two Bell states which are maximally entangled :

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) \quad \text{and} \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle)$$

Logical Bell inequalities

Suppose we have N logic propositional formulas A_1, \dots, A_N . We suppose further that we can assign a probability P_i to each proposition, the probabilities $P_i = P(A_i)$ are obtained from the statistics of the experiments.

Let $P(\mathcal{A})$ be the probability of the joint global event $\mathcal{A} = \bigwedge_{i=1}^N A_i$.

Then using the rule of negation/complementation, De Morgan laws and Boole's inequality:

$$1 - P(\mathcal{A}) = P(\neg \mathcal{A}) = P(\neg \bigwedge_{i=1}^N A_i) = P(\bigvee_{i=1}^N \neg A_i) \leq \sum_{i=1}^N P(\neg A_i) = \sum_{i=1}^N (1 - P_i) = N - \sum_{i=1}^N P_i$$

giving:

$$\sum_{i=1}^N P_i \leq N + P(\mathcal{A}) - 1$$

If we suppose that the propositional formulas A_i are **jointly contradictory** we have $P(\mathcal{A}) = 0$

Leading to the **logical Bell inequality** [*] $\sum_{i=1}^N P_i \leq N - 1$

For the CHSH scenario with $N = 4$ experiments : $\sum_{i=1}^4 P_i \leq 4 - 1 = 3$

the expectation values $-1 \leq E_i \leq +1$ for the outcomes are given by: $E_i = (+1)P_i + (-1)(1 - P_i) = 2P_i - 1$

leads to $\sum_{i=1}^4 E_i = \sum_{i=1}^4 (2P_i - 1) = 2 \sum_{i=1}^4 P_i - 4 \leq 2(4 - 1) - 4 = 2$

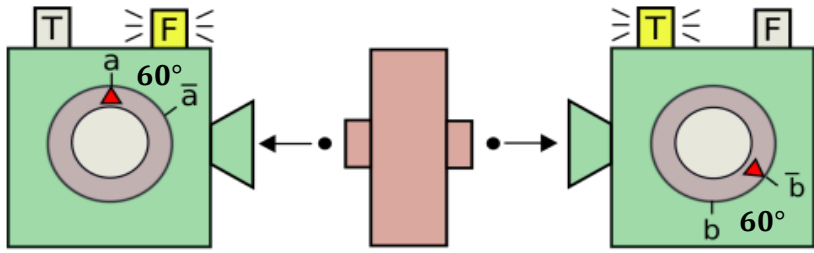
giving $|\sum_{i=1}^4 E_i| \leq 2$ which is the **CHSH inequality**

[*] S. Abramsky and L. Hardy, **Logical Bell inequalities**, Phys. Rev. A 85, 062114 (2012)

CHSH inequalities situations: $\theta = 60^\circ$

The two quantum settings a and a' on the left and b and b' on the right are at $\theta = 60^\circ$ degrees apart respectively.

The respective quantum probabilities are: $P(T,T) = P(F,F) = \frac{1}{2} \cos^2 \frac{\theta}{2}$, $P(T,F) = P(F,T) = \frac{1}{2} \sin^2 \frac{\theta}{2}$



setting	(T,T)	(T,F)	(F,T)	(F,F)
ab	1/2	0	0	1/2
ab'	3/8	1/8	1/8	3/8
$a'b$	3/8	1/8	1/8	3/8
$a'b'$	1/8	3/8	3/8	1/8

The **probabilities in red** correspond to the four logical propositions which are conjointly contradictory:

$$(a \wedge b) \vee (\neg a \wedge \neg b) = a \leftrightarrow b$$

$$(a \wedge b') \vee (\neg a \wedge \neg b') = a \leftrightarrow b'$$

$$(a' \wedge b) \vee (\neg a' \wedge \neg b) = a' \leftrightarrow b$$

$$(\neg a' \wedge b') \vee (a' \wedge \neg b') = a' \oplus b' = \neg(a' \leftrightarrow b')$$

The logical Bell inequality $\sum_{i=1}^4 P_i = 2 \times \frac{1}{2} + 3 \times \frac{3}{4} = 3.25 > 3$ is violated.

For the outcomes we have $E(a, b) = 2P_1 - 1 = 1 = E_1$, $E(a, b') = 2P_2 - 1 = \frac{3}{2} - 1 = \frac{1}{2} = E_2 = E_3 = E(a', b)$

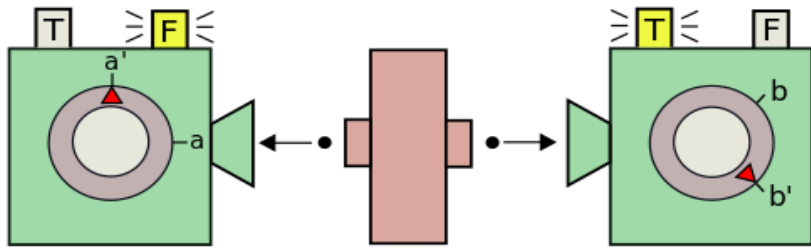
but for $E(a', b') = 1 - 2P_4 = -\frac{1}{2} = -E_4$ (this is where the minus sign in the CHSH inequality comes from !)

the CHSH inequality $E(a, b) + E(a, b') + E(a', b) - E(a', b') = \sum_{i=1}^4 E_i = 1 + 3 \times \frac{1}{2} = 2.5$ **is violated**

CHSH inequalities situations: $\theta = 45^\circ$

Two quantum settings a and a' and b and b' at right angles and right setting rotated by $\theta = 45^\circ$ degrees from left.

The respective quantum probabilities are: $P(T, T) = P(F, F) = \frac{1}{2} \cos^2 \frac{\theta}{2}$, $P(T, F) = P(F, T) = \frac{1}{2} \sin^2 \frac{\theta}{2}$



setting	(T,T)	(T,F)	(F,T)	(F,F)
ab	$\frac{1}{8}(2 + \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 + \sqrt{2})$
ab'	$\frac{1}{8}(2 + \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 + \sqrt{2})$
$a'b$	$\frac{1}{8}(2 + \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 + \sqrt{2})$
$a'b'$	$\frac{1}{8}(2 - \sqrt{2})$	$\frac{1}{8}(2 + \sqrt{2})$	$\frac{1}{8}(2 + \sqrt{2})$	$\frac{1}{8}(2 - \sqrt{2})$

The **probabilities in red** correspond to the four logical propositions which are conjointly contradictory:

$$(a \wedge b) \vee (\neg a \wedge \neg b) = a \leftrightarrow b$$

$$(a \wedge b') \vee (\neg a \wedge \neg b') = a \leftrightarrow b'$$

$$(a' \wedge b) \vee (\neg a' \wedge \neg b) = a' \leftrightarrow b$$

$$(\neg a' \wedge b') \vee (a' \wedge \neg b') = a' \oplus b' = \neg(a' \leftrightarrow b')$$

The logical Bell inequality $\sum_{i=1}^4 P_i = 2 + \sqrt{2} = 3.414 > 3$ is violated

$$\text{For the outcomes we have } E(a, b) = 2P_1 - 1 = E_1 = \frac{1}{2}(2 + \sqrt{2}) - 1 = \frac{\sqrt{2}}{2} = E(a, b') = E_2 = E(a', b) = E_3$$

$$\text{on the other hand for } E(a', b') = 1 - 2P_4 = -E_4 = -\frac{\sqrt{2}}{2}$$

the CHSH inequality :

$$E(a, b) + E(a, b') + E(a', b) - E(a', b') = \sum_{i=1}^4 E_i = 4 \times \frac{\sqrt{2}}{2} = 2\sqrt{2} = 2.8284 \quad \text{maximal quantum violation}$$

The CHSH Bell inequality and Eigenlogic

The CHSH Bell inequality is expressed with the Pauli spin operators σ_i along the 4 measurement directions

The CHSH measurement operator is then :
$$\mathbf{S} = \sigma_A \otimes \sigma_B + \sigma_A \otimes \sigma_{B'} + \sigma_{A'} \otimes \sigma_B - \sigma_{A'} \otimes \sigma_{B'}$$

Considering the projection operators, each term transforms as:
$$\sigma_A \otimes \sigma_B = (\mathbb{I} - 2\Pi_A) \otimes (\mathbb{I} - 2\Pi_B)$$
 replacing and simplifying :

$$\mathbf{S} = 2\mathbb{I} - 4\Pi_A \otimes \mathbb{I} - 4\mathbb{I} \otimes \Pi_B + 4\Pi_A \otimes \Pi_B + 4\Pi_A \otimes \Pi_{B'} + 4\Pi_{A'} \otimes \Pi_B - 4\Pi_{A'} \otimes \Pi_{B'}$$

in this expression one recognizes the Eigenlogic projection and conjunction operators

To evaluate the Bell inequality parameter S one averages this operator:
$$S = \langle \psi | \mathbf{S} | \psi \rangle$$

By averaging the operator $\mathcal{F} = \frac{1}{4}\mathbf{S} - \frac{\mathbb{I}}{2}$ one obtains the Fine inequality for probabilities:

$$\mathcal{F} = \langle \psi | \mathcal{F} | \psi \rangle = P(A \wedge B) + P(A \wedge B') + P(A' \wedge B) - P(A' \wedge B') - P(A) - P(A) = \frac{1}{4}(S - 2)$$

classically $-1 \leq \mathcal{F} \leq 0$ equivalent to $-2 \leq S \leq +2$ for entangled states one has violation of these inequalities.

George Boole already discussed these probability inequalities in 1854 as stated by **Itamar Pitowsky** in [*]

[*] Pitowsky I. **From George Boole To John Bell — The Origins of Bell's Inequality.** In: Kafatos M. (eds) Bell's Theorem, Quantum Theory and Conceptions of the Universe. Fundamental Theories of Physics, vol 37. Springer (1989)

Logical Boxes

the Bell CHSH inequality cases

Classical, local, separable

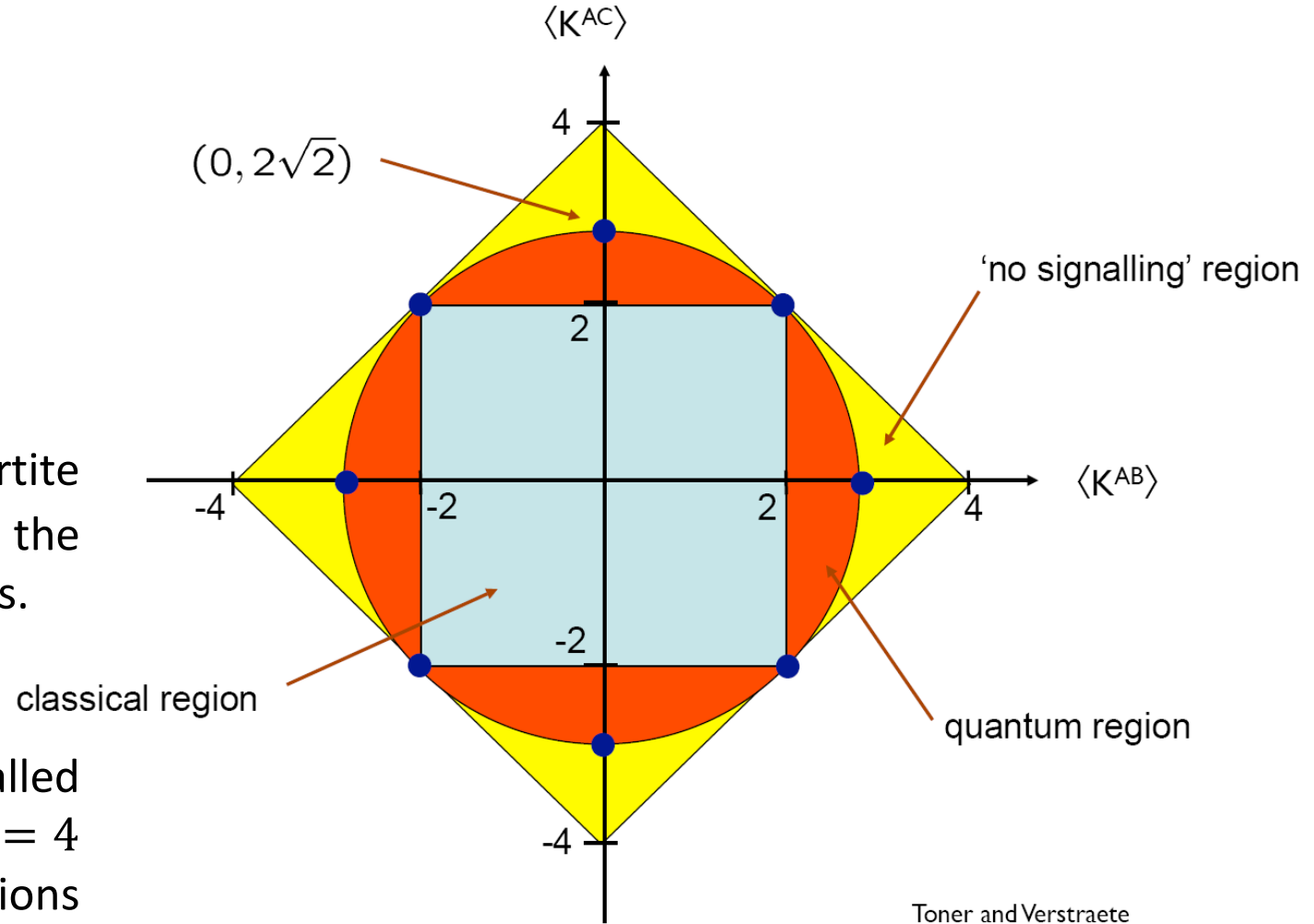
The Bell parameter S_{Bell} lies between 0 and 2.
Measurements are local: $E(X, Y) = E(X)E(Y)$.

Quantum

The case $2 \leq S_{Bell} \leq 2\sqrt{2}$ achieved with bipartite quantum entangled states. $S_{Bell} = 2\sqrt{2}$ is called the Tsirelson's bound and is a limit for quantum systems.

Post-quantum

The case between $2\sqrt{2}$ and 4 comprises the so-called “no-signalling” region. The maximum value $S_{Bell} = 4$ can be attained with logical probabilistic constructions often named **non-local boxes**.



the strange propositions of Diederik Aerts [*]

D. Aerts in 1982 proposed a **macroscopic experiment** that violates the CHSH Bell Inequality maximally.

Two vessels V1 and V2 with a capacity of 8 liters each, linked through a tube with a capacity of 16 liters (at most the system holds 32 liters). The vessel Vref used to siphon water from the V1 and V2 basins.

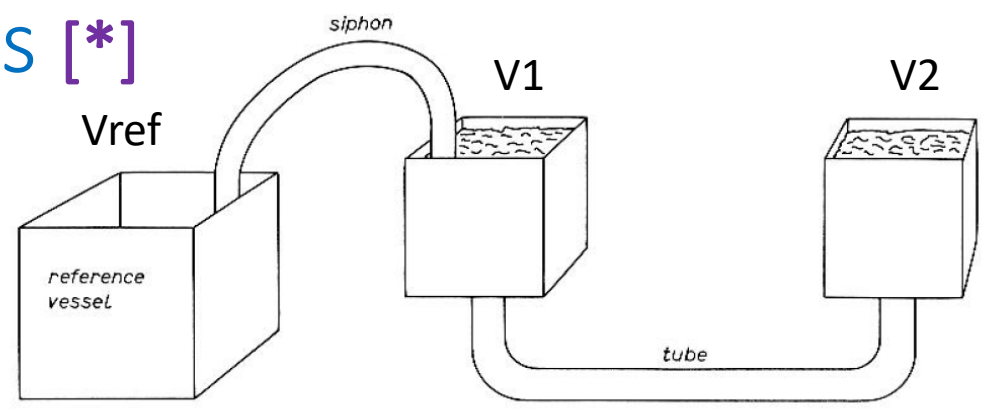


Fig. 1.

4 experiments :

- Experiment α :** answer to: *is $V_{ref} > 10 L$?*
- Experiment β :** answer to: *are V1 or V2 > 6L ?*
- Experiment γ :** answer to: *is the water drinkable ?*
- Experiment δ :** answer to: *is the water transparent ?*

[*] D. Aerts, **Example of a macroscopical situation that violates Bell inequalities**, Lett. Nuovo Cimento **34**, 107 (1982)

outcomes: +1 if the answer is YES and -1 if the answer is NO

Results of correlated experiments:
 $X_{\alpha,\beta} = -1$, $X_{\alpha,\gamma} = +1$, $X_{\delta,\beta} = +1$ and $X_{\delta,\gamma} = +1$.

Taking the sum for the CHSH Bell parameter:

$S = |X_{\alpha,\beta} - X_{\alpha,\gamma}| + |X_{\delta,\beta} + X_{\delta,\gamma}| = 4$
Bell's inequality is therefore maximally violated!



In 2012 we undertook the experiment at Supélec using 2 flower pots and a 32 m water tube. students: Vincent DUMOULIN & Yves SOURRILLE

the logic of a PR Box

The well known nonlocal PR box [*] correlates outputs (a, b) to inputs (x, y) in a two-party correlation by means of a logical constraint equation:

$$a \oplus b \leftrightarrow x \wedge y$$

This box violates the CHSH Bell Inequality (BI) maximally. The measurement outcomes (A, B) , Alice and Bob, give the values ± 1 .

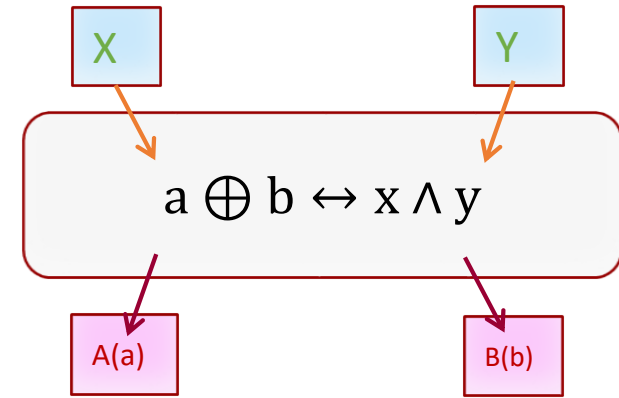
We define the joint mean value for the possible outcomes of the box as a function of the marginal probabilities :

$$C_{x,y} = \sum_{a,b} P(a, b | a \oplus b \leftrightarrow x \wedge y) \cdot A(a) \cdot B(b)$$

where $A(a) = 1 - 2a = (-1)^a$; $B(b) = 1 - 2b = (-1)^b$

The Bell parameter considering the four input possibilities is:

$$\begin{aligned} S &= \sum_{a,b,x,y} (-1)^{a+b+xy} P(a, b | x, y) = \\ &= C_{00} + C_{01} + C_{10} - C_{11} = 4 \end{aligned}$$



x	0	0	1	1
y	0	1	0	1
$x \wedge y$	0	0	0	1
C_{xy}	+1	+1	+1	-1
$a \oplus b$	0	1	1	0
a , A	0 +1	0 +1	1 -1	1 -1
b , B	0 +1	1 -1	0 +1	1 -1

$$C_{xy} = P(0,0|x,y) \cdot (+1) \cdot (+1) + P(0,1|x,y) \cdot (+1) \cdot (-1) + P(1,0|x,y) \cdot (-1) \cdot (+1) + P(1,1|x,y) \cdot (-1) \cdot (-1)$$

$$C_{00} = C_{01} = C_{10} = \frac{1}{2} + 0 + 0 + \frac{1}{2} = +1$$

$$C_{11} = 0 - \frac{1}{2} - \frac{1}{2} + 0 = -1$$

[*] Popescu S., Rohrlich D. **Quantum nonlocality as an axiom.** Found Phys 24, 379–385 (1994)

CHSH inequalities situations: PR box

Settings : a, a', b, b'

the respective probabilities are:

setting	(T,T)	(T,F)	(F,T)	(F,F)
ab	1/2	0	0	1/2
ab'	1/2	0	0	1/2
$a'b$	1/2	0	0	1/2
$a'b'$	0	1/2	1/2	0

The **probabilities in red** correspond to the four logical propositions which are conjointly contradictory:

$$(a \wedge b) \vee (\neg a \wedge \neg b) = a \leftrightarrow b$$

$$(a \wedge b') \vee (\neg a \wedge \neg b') = a \leftrightarrow b'$$

$$(a' \wedge b) \vee (\neg a' \wedge \neg b) = a' \leftrightarrow b$$

$$(\neg a' \wedge b') \vee (a' \wedge \neg b') = a' \oplus b' = \neg(a' \leftrightarrow b')$$

The logical Bell inequality $\sum_{i=1}^4 P_i = 4 > 3$ is violated.

For the outcomes we have $E(a, b) = 2P_1 - 1 = 1 = E_1 = E(a', b) = E_2 = E(a, b') = E_3$

and for $E(a', b') = 1 - 2P_4 = -1 = -E_4$

The CHSH inequality $E(a, b) + E(a, b') + E(a', b) - E(a', b') = \sum_{i=1}^4 E_i = 4$ **maximal CHSH violation**

analysing the PR Box Bell inequality by Eigenlogic

One uses the logical expression directly in an operator form using the following logical identity on the equivalence connective \leftrightarrow leading to the Reed-Muller form:

$$a \oplus b \leftrightarrow x \wedge y = \overline{a \oplus b \oplus x \wedge y}$$

Using the involution properties:

$$(-1)^{\overline{a \oplus b \oplus x \wedge y}} = -(-1)^{a \oplus b \oplus x \wedge y} = -(-1)^a (-1)^b (-1)^{x \wedge y}$$

One can then express the involution \mathbf{G}_{PR} operator's eigenvalues (truth values) by:

$$-(-1)^a (-1)^b (-1)^{xy} = -(-1)^{+a+b+xy}$$

The corresponding Eigenlogic projective operator is:

$$\mathbf{F}_{\text{PR}} = \mathbf{F}_{a \oplus b \leftrightarrow x \wedge y} = \frac{1}{2} (\mathbb{I} - \mathbf{G}_{\text{PR}})$$

The Bell parameter S is obtained by averaging the operator \mathbf{G}_{PR}

for all the cases verifying the logical constraint $a \oplus b \leftrightarrow x \wedge y$

that is to the 8 cases out of 16 where the truth value of \mathbf{F}_{PR} is 1 .

The truth value 0 corresponding to the other 8 cases where $a \oplus b \leftrightarrow x \wedge y$ is not satisfied.

Using: $\mathbf{G}_{\text{PR}} = \mathbb{I} - 2\mathbf{F}_{\text{PR}}$ and the idempotence : $\mathbf{F}_{\text{PR}}^2 = \mathbf{F}_{\text{PR}}$

$$S = \frac{8}{16} |\text{Tr}(\mathbf{F}_{\text{PR}} \cdot \mathbf{G}_{\text{PR}})| = \frac{1}{2} |\text{Tr}(\mathbf{F}_{\text{PR}}(\mathbb{I} - 2\mathbf{F}_{\text{PR}}))| = \frac{1}{2} \text{Tr}(\mathbf{F}_{\text{PR}}) = \frac{8}{2} = 4$$

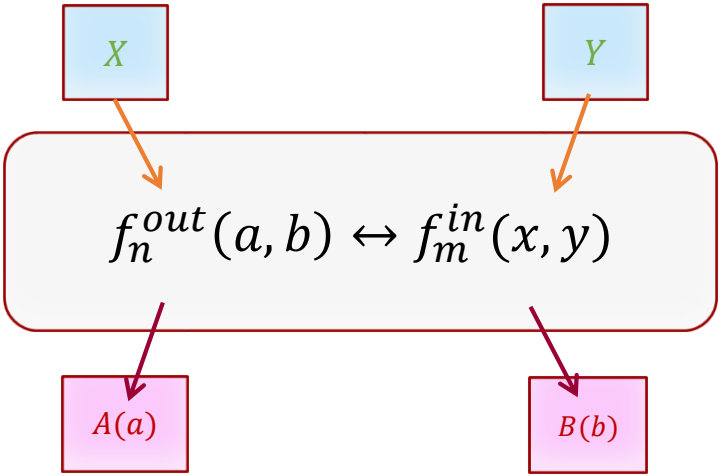
Generalizing the PR Box BI for all logical bipartite constraints

All combinations produce $16 \times 16 = 256$ logical constraint boxes.

The PR box corresponds to: $a \oplus b \leftrightarrow x \wedge y$ ($f_6(a, b) \leftrightarrow f_8(x, y)$)
BI violation $|S| = 4$ for 16 no-signaling nonlocal boxes (orange boxes)

Other 32 boxes (green boxes) are signaling and violate BI with $|S| = \frac{10}{3} \approx 3.33$

is the case for: $a \vee b \leftrightarrow x \wedge y$ ($f_{14}(a, b) \leftrightarrow f_8(x, y)$)



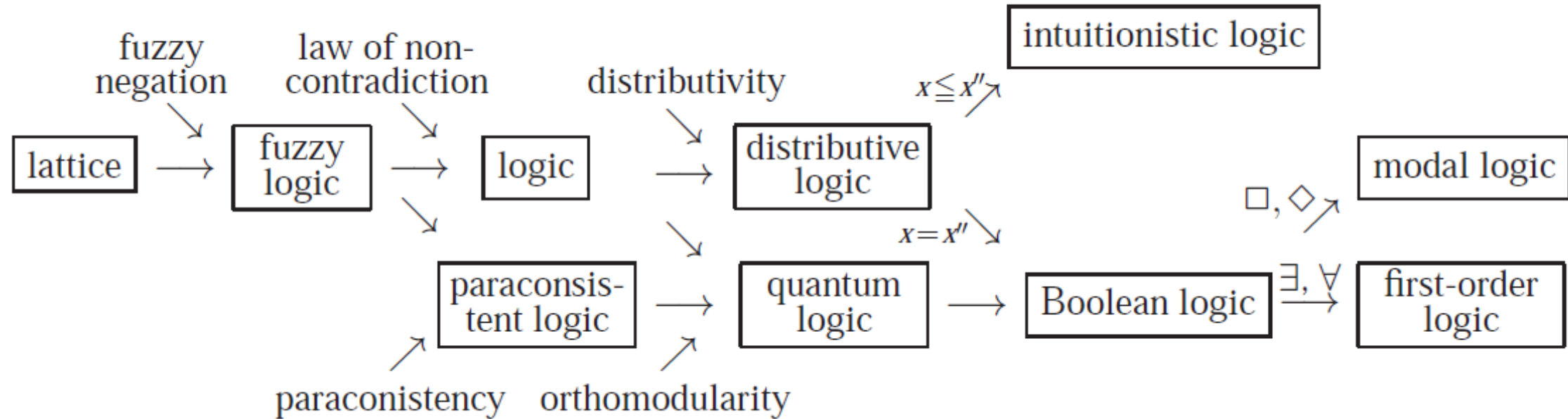
Logical function for p, q $f_n(p, q)$	Truth table for 2 inputs				Logical operator
	(p, q)	(1,1)	(1,0)	(0,1)(0,0)	
f_0	(0	0	0	0)	contradiction : False ; \perp
f_1	(0	0	0	1)	not or : NOR ; $\neg P \wedge \neg Q$
f_2	(0	0	1	0)	non-implication : $P \supseteq Q$
f_3	(0	0	1	1)	negation of p : $\neg P$
f_4	(0	1	0	0)	converse non-implication : $P \subseteq Q$
f_5	(0	1	0	1)	negation of q : $\neg Q$
f_6	(0	1	1	0)	exclusive or : XOR ; $P \oplus Q$
f_7	(0	1	1	1)	not and : NAND ; $\neg P \vee \neg Q$
f_8	(1	0	0	0)	conjunction (and) : AND ; $P \wedge Q$
f_9	(1	0	0	1)	equivalence : XNOR ; $P \equiv Q$
f_{10}	(1	0	1	0)	right projection of q : Q
f_{11}	(1	0	1	1)	converse implication $P \subset Q$
f_{12}	(1	1	0	0)	left projection of p : P
f_{13}	(1	1	0	1)	implication : $P \supset Q$
f_{14}	(1	1	1	0)	disjunction (or) : OR ; $P \vee Q$
f_{15}	(1	1	1	1)	tautology : True ; T

OUTPUT INPUT	f_0	f_1	f_2	f_4	f_8	f_3	f_5	f_{12}	f_{10}	f_6	f_9	f_7	f_{11}	f_{13}	f_{14}	f_{15}
f_0																
f_1										4	4	3,33	3,33	3,33	3,33	
f_2										4	4	3,33	3,33	3,33	3,33	
f_4										4	4	3,33	3,33	3,33	3,33	
f_8										4	4	3,33	3,33	3,33	3,33	
f_3																
f_5																
f_{12}																
f_{10}																
f_6																
f_9																
f_7		3,33	3,33	3,33	3,33					4	4					
f_{11}		3,33	3,33	3,33	3,33					4	4					
f_{13}		3,33	3,33	3,33	3,33					4	4					
f_{14}		3,33	3,33	3,33	3,33					4	4					
f_{15}																

The Eigenlogic program

- ❑ Multivalued Eigenlogic
- ❑ Fuzzy Eigenlogic
- ❑ Quantum computing in Eigenlogic
- ❑ Towards first-order Eigenlogic
- ❑ Quantum-like combinatory logic
- ❑ ...

trying to classify Eigenlogic



Following the classification of the algebraic hierarchy of logics proposed **Andreas de Vries** [*].

Eigenlogic encompassing Boolean, many-valued, fuzzy, quantum logic and first order logic could fit into this diagram

[*] A. de Vries, **Algebraic hierarchy of logics unifying fuzzy logic and quantum logic**, arXiv:0707.2161 (2007) and in **Quantum Computation**, chap 13 **Quantum Logic** Ed. Books on Demand (2012)

Multivalued Eigenlogic of quantum angular momentum

balanced ternary logic

for **Orbital Angular Momentum (OAM)** with $\ell = 1$.

The z component of the orbital angular momentum operator :

$$L_z = \hbar \Lambda = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hbar \text{diag}(+1, 0, -1)$$

the three rank-1 projectors obtained by interpolation :

$$\Pi_{+1} = \frac{1}{2} \Lambda (\Lambda + \mathbb{I}) \quad , \quad \Pi_0 = \mathbb{I} - \Lambda^2 \quad , \quad \Pi_{-1} = \frac{1}{2} \Lambda (\Lambda - \mathbb{I})$$

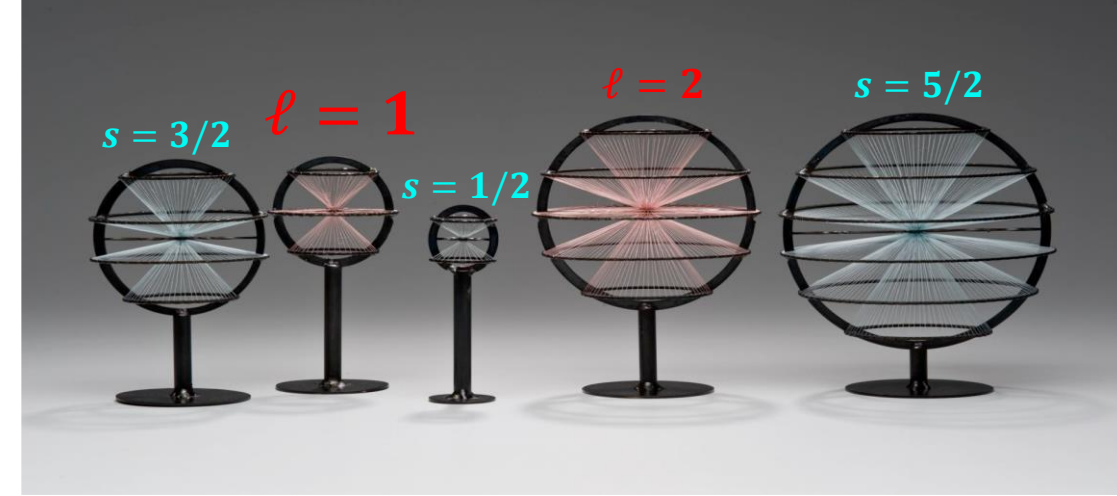
For arity-2 U and V are then defined as usual in Eigenlogic :

$$U = \Lambda \otimes \mathbb{I} \quad , \quad V = \mathbb{I} \otimes \Lambda$$

In many-valued logic the Min and Max are the equivalent of AND and OR :

$$\text{Min}(U, V) = \frac{1}{2} (U + V + U^2 + V^2 - U \cdot V - U^2 \cdot V^2) = \text{diag}(1, 1, 1, 1, 0, 0, 1, 0, -1)$$

$$\text{Max}(U, V) = \frac{1}{2} (U + V - U^2 - V^2 + U \cdot V + U^2 \cdot V^2) = \text{diag}(1, 0, -1, 0, 0, -1, -1, -1, -1)$$



Spin Family (Bosons ℓ and Fermions s)
 (© Julian Voss-Andreae. Photo: Dan Kvitka.)

<i>Min</i> $U \backslash V$	F	N	T
False $\equiv +1$	+1	+1	+1
Neutral $\equiv 0$	+1	0	0
True $\equiv -1$	+1	0	-1

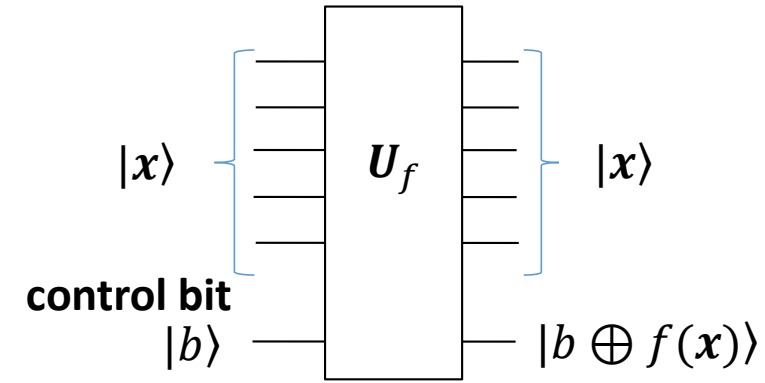
<i>Max</i> $U \backslash V$	F	N	T
False $\equiv +1$	+1	0	-1
Neutral $\equiv 0$	0	0	-1
True $\equiv -1$	-1	-1	-1

expressing a quantum computing oracle circuit in Eigenlogic

A Boolean logical function f is represented by a **quantum oracle** U_f :

Particular cases are:

the 2-qubit CNOT gate C (f is the NOT function)
 the 3-qubit Toffoli gate TO (f is the AND function)



The logical function f is represented by the projective Eigenlogic operator F

The control bit corresponds to the seed projection operator in the $|x\rangle$ basis: $\Pi_- = |-\rangle\langle-| = \frac{1}{2}(\mathbb{I} - X)$

The oracle is then simply expressed in Eigenlogic as :
 (similar approach by **S. Hadfield** in **[*]**)

$$U_f = (-1)^{F \otimes \Pi_-} = e^{i\pi F \otimes \Pi_-} = \mathbb{I} - 2F \otimes \Pi_-$$

In the case of a one bit Boolean function $f(x)$:

$$U_f = \Pi_0 \otimes X^{f(0)} + \Pi_1 \otimes X^{f(1)}$$

the **Deutsch algorithm** result is obtained by applying the oracle on : $|+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$U_f |+\rangle|-\rangle = \frac{1}{\sqrt{2}} |0\rangle (-1)^{f(0)} |-\rangle + \frac{1}{\sqrt{2}} |1\rangle (-1)^{f(1)} |-\rangle$$

for **constant** $f(0) = f(1)$ $U_f |+\rangle|-\rangle = \pm |+\rangle|-\rangle$ and for **balanced** $f(0) \neq f(1)$ $U_f |+\rangle|-\rangle = \pm |-\rangle|-\rangle$

[*] S. Hadfield, **On the Representation of Boolean and Real Functions as Hamiltonians for Quantum Computing.**

towards first-order Eigenlogic

Using two maximally incompatible logical families with logical eigensystems associated to the X and Z gates (resp. the σ_x and σ_z Pauli operators) one gets an interesting outlook:

the quantum Grover amplification gate used in the *Grover algorithm*, corresponds to the multi-qubit involution Eigenlogic negated disjunction operator NOR in the X system.

This operator can be interpreted in the Z system as a predicative logical *existential connective* \exists .

In the language of first order logic with a 3-qubit phase oracle the Grover circuit operates the following logical proposition:

$$\exists a P(a) \equiv \neg(P_X \vee Q_X \vee R_X) [P_Z \wedge Q_Z \wedge R_Z]$$

A justification can be found in **Herbrand's fundamental theorem** [*] that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.

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[*] J. Herbrand, **Recherches sur la théorie de la démonstration**, Thèses présentées à la faculté des sciences de Paris, Paris, (1930).

Grover algorithm and first-order-logic

The **Grover search algorithm** looks for an element a (here $|a_2 a_1 a_0\rangle = |111\rangle$) satisfying the property P (**oracle**) and becomes the predicate proposition in **first-order-logic** using the **existential logical quantifier** \exists :

$$\exists a P(a)$$

The **Grover amplification gate** corresponds to an Eigenlogic negated disjunction operator NOR in the X system.

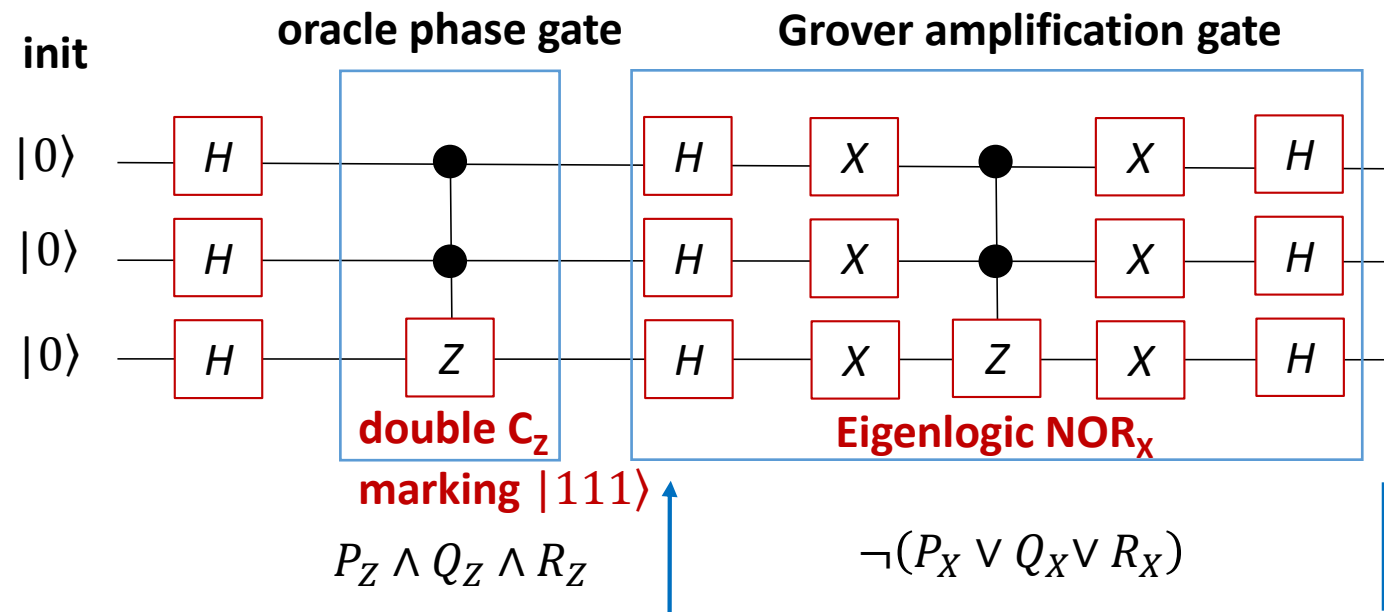
The **phase oracle** is a **double control Z gate** (Eigenlogic 3-input AND: $(-1)^{\Pi \otimes \Pi \otimes \Pi}$).

$\exists a P(a)$ decomposes, using **Skolemization** methods into a succession of “disjunction \vee ” connectives.

$$\exists a \Leftrightarrow a_1 \vee a_2 \vee \dots \vee a_N$$

$$\forall a \Leftrightarrow a_1 \wedge a_2 \wedge \dots \wedge a_N$$

A justification can be found in **Herbrand’s theorem** that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.



quantum-like Combinatory Logic

Moses Schönfinkel [a] introduced a method in logic named *Combinatory Logic*, this was part of the Hilbert program aimed to formulate all the fields of mathematics in a consistent logic system.

Haskell Curry successively improved and completed the research [b]. This led to the development of *functional programming languages* such as *Haskell*, and *Erlang*.

Combinatory logic uses abstract operators (**combinators**) to compose and to transform operators and arguments.

It permits to translate first order logic into expressions without variables using only combinators without the need of the universal quantifier \forall and the existential quantifier \exists .

Alessandra di Pierro, in [c] considers that “...reversible combinatory logic can in principle be used for a ... translation of classical into quantum computation.”

A tentative approach could consist in identifying the different operations of substitution, elimination, permutation, etc., with equivalent operations obtained using quantum gates.

tricks in q. computation could be used: $C_Z \cdot (X \otimes Z) \cdot C_Z = X \otimes \mathbb{I}_2$ or $U_{\text{swap}} \cdot (P \otimes Q) \cdot U_{\text{swap}} = Q \otimes P$

[a] Schönfinkel, M. **Über die Bausteine der mathematischen Logik**. Math. Ann, 92, 305–316. (1924)

[b] H.B. Curry, **Combinatory Logic**, North-Holland Co: Amsterdam, (1958)

[c] A. Di Pierro, **On Reversible Combinatory Logic**, Electron. Notes Theor. Comput. Sci. 135, 25–35 (2006)

Qubits and quantum circuits

Eigenlogic and 2-qubit quantum gates

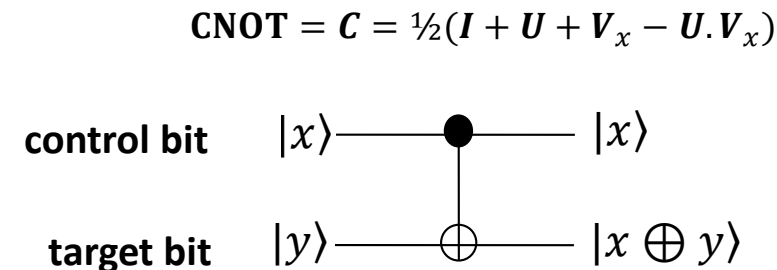
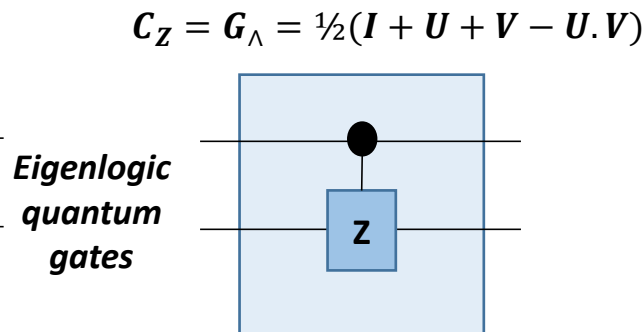
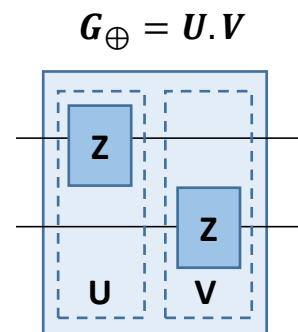
Eigenlogic makes a correspondence between **quantum control logic** (David Deutsch's quantum logical gate paradigm) and ordinary propositional logic.

It is known that the 2-qubit **control-phase gate** C_Z in association with 1-qubit gates is a **universal gate set**.

In Eigenlogic the C_Z gate corresponds to the AND involution gate G_\wedge : $C_Z = G_\wedge = \text{diag}(1,1,1,-1)$

The well-known control-not **CNOT gate** C can be expressed using the Pauli matrices $\sigma_z = Z$ and $\sigma_x = X$ using the **Eigenlogic involution conjunction operator** (in the alphabet $\{+1, -1\}$) one has directly:

$$C = (-1)^{\Pi \otimes \Pi_-} = e^{i\pi \Pi \otimes \Pi_-} = \mathbb{I} - 2(\Pi \otimes \Pi_-) = \frac{1}{2}(\mathbb{I} + Z \otimes \mathbb{I} + \mathbb{I} \otimes X - Z \otimes X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



building the 3-qubit Toffoli universal quantum gate

The 3-qubit Toffoli (double-CNOT) gate **TO** is a **universal reversible logic quantum gate**, directly in **Eigenlogic form** :

$$\begin{aligned} \mathbf{TO} &= (-1)^{\Pi \otimes \Pi \otimes \Pi} = \mathbb{I} - 2(\Pi \otimes \Pi \otimes \Pi) \\ &= \frac{1}{4}(3 \mathbb{I} + \mathbf{Z}_2 + \mathbf{Z}_1 + \mathbf{X}_0 - \mathbf{Z}_2 \cdot \mathbf{Z}_1 - \mathbf{Z}_2 \cdot \mathbf{X}_0 - \mathbf{Z}_1 \cdot \mathbf{X}_0 + \mathbf{Z}_2 \cdot \mathbf{Z}_1 \cdot \mathbf{X}_0) \end{aligned}$$

Can be put in exponential form using the Householder transform

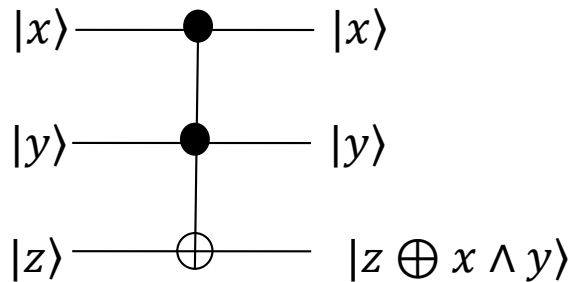
$$\mathbf{TO} = e^{+i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\mathbf{Z}_1} e^{-i\frac{\pi}{8}\mathbf{Z}_2} e^{-i\frac{\pi}{8}\mathbf{X}_0} e^{i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{Z}_1} e^{i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{X}_0} e^{i\frac{\pi}{8}\mathbf{Z}_1 \cdot \mathbf{X}_0} e^{-i\frac{\pi}{8}\mathbf{Z}_2 \cdot \mathbf{Z}_1 \cdot \mathbf{X}_0}$$

Alternative method using a T gate as the Eigenlogic seed operator: $\mathbf{T} = \mathbf{Z}^{\frac{1}{4}} = e^{i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\mathbf{Z}} = \text{diag}\left(1, e^{i\frac{\pi}{4}}\right)$

using a **Reed-Muller form** for the CCZ gate [*]: $\mathbf{C}_{CZ} = \mathbf{T}_0 \cdot \mathbf{T}_1 \cdot \mathbf{T}_2 \cdot (\mathbf{T}_{x \oplus y})^\dagger \cdot (\mathbf{T}_{x \oplus z})^\dagger \cdot (\mathbf{T}_{y \oplus z})^\dagger \cdot (\mathbf{T}_{x \oplus y \oplus z})$

by the **Hadamard gate** extension one has again the Toffoli gate :

$$\mathbf{TO} = \mathbf{H}_0 \cdot \mathbf{C}_{CZ} \cdot \mathbf{H}_0$$



control bit

$$\mathbf{TO} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

[*] Selinger, P., **Quantum circuits of T-depth one**, Phys. Rev. A, 87, 252–259, (2013)

logical universality and quantum entanglement

In binary propositional logic the following set of **8 connectives** when combined with negation (NOT) constitutes

a **universal gate set** :

AND, OR, NOR, NAND, \Rightarrow , \nRightarrow , \Leftarrow , \nLeftarrow

one observes that they possess an **odd number of True and False truth values**

the other 8 connectives are not universal:

P, Q, $\neg P$, $\neg Q$, \equiv , XOR, F ,T

one observes that they possess an **even number of True and False truth values**

For involution logical operators G with eigenvalues $\{+1, -1\}$

the universal logical gates correspond to 8 operators with an odd number of eigenvalues $+1$ and -1

therefore these are all entangling gates

the 8 other logical operators with even number of eigenvalues are separable (not entangled) and not universal.

This states clearly the correspondence between logical universality and entanglement.

P	Q	F	NOR	$P \nLeftarrow Q$	$\neg P$	$P \nRightarrow Q$	$\neg Q$	XOR	NAND	AND	$P \equiv Q$	Q	$P \Rightarrow Q$	P	$Q \Rightarrow P$	OR	T
+	+	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
+	-	+	+	-	-	-	+	-	-	+	+	-	-	+	+	-	-
-	+	+	+	+	+	+	-	-	-	+	+	+	+	-	-	-	-
-	-	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-

Deutsch algorithm [*]

The Deutsch algorithm is one of the first quantum algorithms more efficient than its classical counterpart.



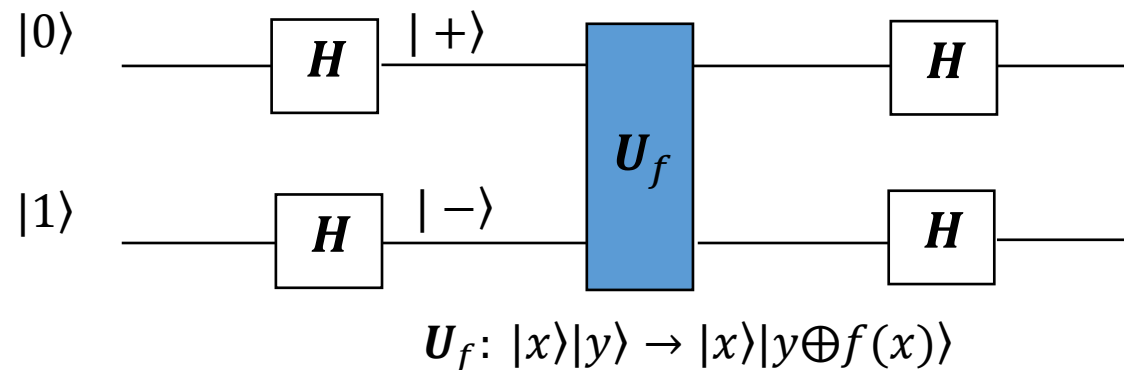
The answer to the question: is the logical function $f(x)$ **constant** or **balanced**?
 can be **performed by a quantum computer in one step.**
 (the classic treatment requires two steps.)

x	$f_{00}(x)$	$f_{01}(x)$	$f_{10}(x)$	$f_{11}(x)$
0	0	0	1	1
1	0	1	0	1

↑
↑
↑
↑
constant
balanced
balanced
constant

[*] D. Deutsch, **Quantum theory, the Church-Turing principle and the universal quantum computer**, Proc. R. Soc. A, 400, 97–117, (1985)

The algorithm measurement is made on the upper qubit of the following circuit



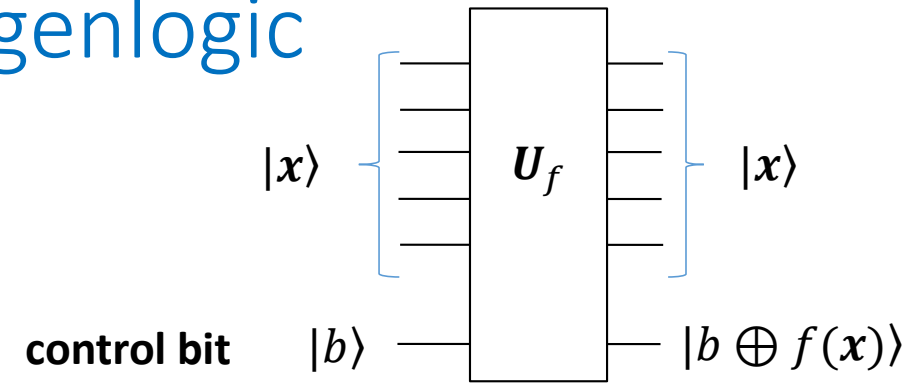
$$\begin{aligned}
 &= (-1)^{f(0)} |f(0) \oplus f(1)\rangle |1\rangle \\
 &= (-1)^{f(0)} |0\rangle |1\rangle \text{ if constant} \\
 &= (-1)^{f(0)} |1\rangle |1\rangle \text{ if balanced}
 \end{aligned}$$

expressing the quantum oracle circuit in Eigenlogic

A Boolean logical function f is represented by a **quantum oracle** U_f :

Particular cases are:

the 2-qubit CNOT gate \mathbf{C} (f is the NOT function)
 the 3-qubit Toffoli gate \mathbf{TO} (f is the AND function)



The logical function f is represented by the projective Eigenlogic operator \mathbf{F}

The control bit corresponds to the seed projection operator in the $|x\rangle$ basis: $\mathbf{\Pi}_- = |-\rangle\langle-| = \frac{1}{2}(\mathbb{I} - \mathbf{X})$

The oracle is then simply expressed in Eigenlogic as :
 (similar approach by **S. Hadfield** in **[*]**)

$$U_f = (-1)^{F \otimes \Pi_-} = e^{i\pi F \otimes \Pi_-} = \mathbb{I} - 2F \otimes \Pi_-$$

In the case of a one bit Boolean function $f(x)$:

$$U_f = \mathbf{\Pi}_0 \otimes \mathbf{X}^{f(0)} + \mathbf{\Pi}_1 \otimes \mathbf{X}^{f(1)}$$

the **Deutsch algorithm** result is obtained by applying the oracle on : $|+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$U_f |+\rangle|-\rangle = \frac{1}{\sqrt{2}} |0\rangle (-1)^{f(0)} |-\rangle + \frac{1}{\sqrt{2}} |1\rangle (-1)^{f(1)} |-\rangle$$

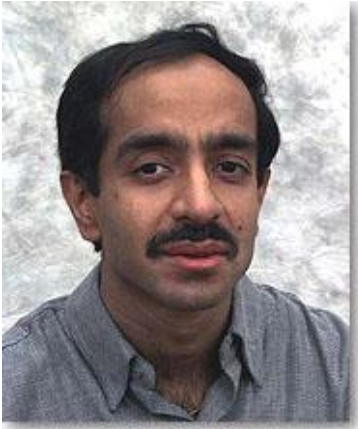
for **constant** $f(0) = f(1)$ $U_f |+\rangle|-\rangle = \pm |+\rangle|-\rangle$ and for **balanced** $f(0) \neq f(1)$ $U_f |+\rangle|-\rangle = \pm |-\rangle|-\rangle$

[*] S. Hadfield, **On the Representation of Boolean and Real Functions as Hamiltonians for Quantum Computing.**

to appear in ACM Transactions on Quantum Computing, arXiv:1804.09130

Grover's search algorithm [*]

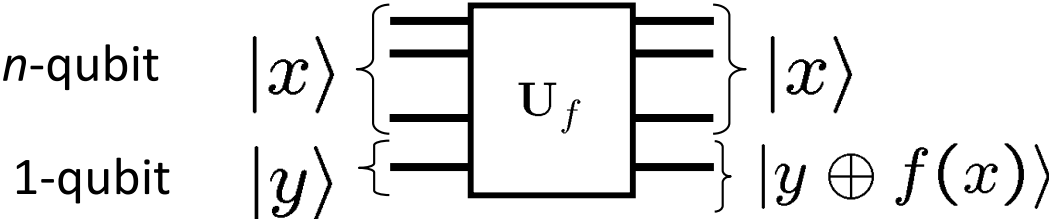
Problem: finding the value x_0 in a large database with the fewest queries possible.



We consider a “black box” (oracle U_f)

Having for the value x_0 the property :

$$f(x_0) = 1 \text{ and } f(x) = 0 \text{ for } x \neq x_0$$



[*] L. K. Grover. **A fast quantum mechanical algorithm for database search**, Proceedings, 28th Annual ACM Symp. on the Theory of Computing, p. 212, (1996)

Classical query:	complexity: $O(\exp(n))$
Grover quantum query:	complexity: $O(\exp(\sqrt{n}))$

Grover algorithm and first-order-logic

The **Grover search algorithm** looks for an element a (here $|a_2 a_1 a_0\rangle = |111\rangle$) satisfying the property P (**oracle**) and becomes the predicate proposition in **first-order-logic** using the **existential logical quantifier** \exists :

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A justification can be found in **Herbrand’s theorem** that provides a constructive characterization of derivability in first-order predicate logic by means of propositional (sentential) logic.

