On the Spectrum Distribution of Parametric Second-order Delay Differential Equations. Perspectives in Partial Pole Placement

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Abstract

This chapter presents a *parameter*-based frequency-domain approach for the analysis and control of linear time-invariant dynamical systems described by delay-differential equations involving a single pointwise delay. More precisely, the link between the maximal allowable multiplicity of the characteristic roots and the spectral abscissa of the dynamical system is exploited to define appropriate stabilizing controllers. The ideas are explicitly illustrated on a class of second-order dynamical systems where a complete parametric characterization of the multiplicity-induced-dominancy is provided using certified algorithms from the computer algebra system `Maple`.

Keywords. Time-delay, asymptotic stability, exponential stability, exponential decay rate, control design, MID property.

1 Introduction

One of the most intuitive ideas pertaining to the control of the dynamical behavior of the linear time-invariant (LTI) system is to place the corresponding closed-loop poles in some desirable location in the complex plane; the method is called *pole placement* (e.g., [2] and the references therein). The pole placement method relies on two ingredients (i) the perfect knowledge of the state variables; (ii) some appropriate *controllability* assumptions on the system, that is, the possibility to steer a dynamical system from an arbitrary initial state to an arbitrary final state via an apropos set of admissible control laws. If the said method is easy to understand and to apply in the control of finite-dimensional LTI systems, its extension to systems described by delay-differential equations (DDEs) seems to be more involved. More precisely, two issues need to be addressed. First, the introduction of a suitable notion of *controllability* for delay systems, and, second, the in-depth comprehension of the location of the poles of the closed-loop system in terms of the

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controller's parameters, even in all generality, such a question remains open. Yet, several partial results exit; among them, systems with single or multiple constant delays (see for instance [44]); systems with cross-talking delays [37]. For more applications of spectral methods see [24, 36, 39, 40, 46, 47].

For a pole placement to be effective, two robustness requirements are mandatory; the first one is with respect to parametric uncertainties, while the second one is with respect to the digitization purposes. Several pole placement paradigms for the control of LTI delay systems exist, each of which has its advantages and its drawbacks, see for instance [16, 27, 33, 35].

A more recent pole placement analytical paradigm having the advantage of furnishing reduced complexity controllers [3] while guaranteeing a close monitoring of the spectrum migration with respect to parametric changes [32] is called *partial pole placement*. It ensues from an observation of the effect of multiple spectral values on the stability of DDEs, a property called *multiplicity-induced-dominancy* (MID) in [12,29]. Indeed, some works have shown that, for some classes of time-delay systems, a real root of maximal multiplicity is necessarily the rightmost root, a property we call *generic multiplicity-induced-dominancy*, or GMID for short; when it is a multiple root of strictly intermediate over-order multiplicity, it is called *intermediate multiplicity-induced-dominancy* (IMID). This link between maximal multiplicity and dominance of the spectral value is not new. In fact, it has been suggested in [41] after the study of some simple, low-order cases, but without any attempt to address the general case. To the best of the authors' knowledge, very few works have considered this question in more details until recently in publications such as [8, 10, 12–14, 29–31, 42]. The latter studies consider only DDEs with a single delay and show either the IMID or the GMID property for each system under consideration. For instance, the IMID or GMID properties are shown to hold for retarded equations of order 1 in [14], which proves dominance by introducing a factorization of Δ in terms of an integral expression when it admits a root of maximal multiplicity 2; for retarded equations of order 2 with a delayed term of order zero in [13], using also the same factorization technique; or for retarded equations of order 2 with a delayed term of order 1 in [12], where both the IMID and the GMID properties are investigated, using Cauchy's argument principle to prove dominance of the multiple root. Most of these results are actually particular cases of a more general result on the GMID property from [29] for generic retarded DDEs of order n with a delayed "term" (polynomial) of order $n-1$, which relies on ties between quasipolynomials with a real root of maximal multiplicity and the Kummer confluent hypergeometric function in terms of the location of the characteristic roots. Owing to this link, the GMID property has been completely characterized in [10] and extended to neutral DDEs of orders 1 and 2 in [8, 26, 30], as well as to the case of complex conjugate roots of maximal multiplicity in [31]. Also, such an idea has been extended to assign an appropriate number of negative roots rather than assigning a multiple root, allowing to a property called Coexistant Real Roots Inducing Dominancy (CRRID), see for instance [6].

The fact that a spectral value achieves maximum multiplicity imposes algebraic constraints on each of the system's "entries" (polynomial coefficients as well as the delay parameter). An IMID-based approach is proposed in [4] operating the intimate representation of the quasipolynomial to provide conditions for one spectral value with an eligible intermediate multiplicity. This makes it possible to split the system parameters into two categories, some of them considered as model parameters (assumed to be fixed and known) and the remaining ones considered as values to be adjusted. Such a classification opens interesting perspectives in control design, such as the systematic tuning of the gains of the well-known Proportional-Integral-Derivative (PID) controller, able to stabilize single-

input/single-output plants including one delay in the input/output channel, as suggested in [26].

In this chapter, we provide a tutorial on the use of the MID property in the partial pole placement for parametric second-order systems by exploiting the delay as a control parameter. These systems have the lowest order capable of exhibiting an oscillatory response to a step input and provide standard models for the description of a wide range of dynamical processes and, as such, represent a benchmark for the enhancement of the understanding of phenomena appearing in physics, in biology and in engineering. It is widely known that the nature of the corresponding characteristic roots is crucial in the definition of the qualitative behavior of solutions of such systems not only in the open-loop setting but also from a feedback control perspective. In open-loop, second-order systems transfer functions admit two poles which are either both real-valued or form a pair of complex conjugates. In each of such configurations, the MID property suggests an appropriate admissible multiplicity as well as a specific tuning as emphasized in [12]. Since the ensuing study involves a lot of computation and is most easily accomplished either numerically or with the aid of a certified symbolic algebra package, the parametric analysis we present will be carried out using the certified `RootFinding` and `Groebner` packages of the computer algebra system `Maple`.

The remaining of the chapter is organized as follows. Some preliminary results and prerequisites are given in Section 2 where fundamental results of complex analysis as well as some standard theoretical settings in elimination theory are provided. A parametric characterization of the MID approach in second-order systems is carried out in Section 3. Section 4 illustrates the delay effect for the oscillation damping of the standard oscillator. Finally, Section 5 concludes the chapter.

2 Prerequisites

Consider the following class of quasipolynomial functions which correspond to a generic time-delay system with real coefficients a_i, b_i for $i = 1, 2$ and a single delay $\tau > 0$:

$$\Delta(s) = s^2 + a_1\, s + a_0 + (b_1\, s + b_0)\, \mathrm{e}^{-\tau\, s} \tag{2.1}$$

Notice that the quasipolynomial function (2.1) is the characteristic equation of the particular case of the following time-delay system

$$\dot{x}(t) = A\, x(t) + B\, x(t-\tau), \tag{2.2}$$

where $x(t) = (y(t),\, \dot{y}(t))^T \in \mathbb{R}^2$ is the state vector and $A,\, B \in \mathcal{M}_2(\mathbb{R})$ are real-valued matrices given by

$$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -b_0 & -b_1 \end{pmatrix}. \tag{2.3}$$

The notation V^T represents the transpose of V.

The next result plays an important role in the spectral theory of time-delay systems, it enables the construction of an envelope curve around the zeros of the characteristic equation, the details of its proof can be found in [34].

Proposition 2.1. *Let s be a characteristic root of the system* (2.2)*, then, it satisfies*

$$|s| \leq \|A + B\, \mathrm{e}^{-\tau s}\|_2. \tag{2.4}$$

Another fundamental property of complex analysis widely used in the stability analysis of linear time-invariant dynamical systems is the argument principle; see for instance [21, 28].

2.1 The Stépán-Hassard approach

The argument principle establishes a correspondence between the number of zeros minus the number of poles of a meromorphic function F in a simply connected domain $D \subset \mathbb{C}$, and a contour integral, on the boundary ∂D, of the function's logarithmic derivative, which is also the winding number of the curve ∂D.

Several stability methods derive from the argument principle such as the Nyquist criterion and the Mikhaylov curve. The argument principle is also a classical standard means to count the number of unstable roots, see for instance [1]. The said count may also be obtained, in an easier and more elegant way, by the inspection of argument variation. Actually, the combination of the qualitative behavior of both the real $\Re$ and the imaginary $\Im$ parts (seen as real functions in the crossing frequency) of the quasipolynomial function, allows a straightforward application of the Stépán-Hassard formula [22, 45].

The main theorem from [22], which is inspired from [45], emphasizes the link between the number of unstable spectral values $\mathbf{card}(\mathcal{N}_+)$ and the number of critical spectral values $\mathbf{card}(\mathcal{N}_0)$, both taking into account the algebraic multiplicity.

Theorem 2.2 (Hassard, [22], pp. 223)**.** *Consider the linear delay-differential system*

$$\dot{x}(t) = \sum_{l=1}^{L} A_l \, x(t - \tau_l), \tag{2.5}$$

for which the corresponding characteristic equation is defined by the quasipolynomial function:

$$\Delta(s) = \det\left(s\,\mathbb{I} - \sum_{l=1}^{L} \mathrm{e}^{-s\,\tau_l} \, A_l \right). \tag{2.6}$$

Let $\rho_1, \ldots, \rho_r$ be the positive roots of

$$\mathcal{R}(y) = \Re(i^n \, \Delta(i\,y)), \tag{2.7}$$

counting their algebraic multiplicities and ordered so that $0 < \rho_1 \leq \ldots \leq \rho_r$. For each $j = 1, \ldots, r$ such that $\Delta(i\,\rho_j) = 0$, assume that the multiplicity of $i\rho_j$ as a zero of $\Delta(s)$ is the same as the multiplicity of ρ_j as a root of $\mathcal{R}(y)$. Then $\mathbf{card}(\mathcal{N}_+)$ is given by the formula:

$$\mathbf{card}(\mathcal{N}_+) = \frac{n - \mathbf{card}(\mathcal{N}_0)}{2} + \frac{(-1)^r}{2} sgn\, \mathcal{I}^{(\mu)}(0) + \sum_{j=1}^{r} (-1)^{j-1}\, sgn\, \mathcal{I}(\rho_j), \tag{2.8}$$

where μ designates the multiplicity of the zero spectral value of $\Delta(s) = 0$ and

$$\mathcal{I}(y) = \Im(i^{-n} \Delta(iy)). \tag{2.9}$$

Furthermore, $\mathbf{card}(\mathcal{N}_+)$ is odd (respectively, even) if $\Delta^{(\mu)}(0) < 0$ ($\Delta^{(\mu)}(0) > 0$). If $\mathcal{R}(y) = 0$ has no positive zeros, set $r = 0$ and omit the summation term in the expression of $\mathbf{card}(\mathcal{N}_+)$. If $s = 0$ is not a root of the characteristic equation, set $\mu = 0$ and interpret $\mathcal{I}^{(0)}(0)$ as $\mathcal{I}(0)$ and $\Delta^{(0)}(0)$ as $\Delta(0)$.

2.2 Special functions in control design

Quasipolynomials functions admitting multiple roots of maximal multiplicity can be factorized in terms of a confluent hypergeometric function defined hereafter [38].

Definition 2.3. Let $a, b \in \mathbb{C}$ such that $-b \notin \mathbb{N}$, Kummer's confluent hypergeometric function is the entire function $M(a, b, \cdot) : \mathbb{C} \to \mathbb{C}$ defined for $z \in \mathbb{C}$ by the series

$$M(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{2.10}$$

where for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the Pochhammer symbol for the ascending factorial, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$, for $k \in \mathbb{N}$.

The series in (2.10) converges for every $z \in \mathbb{C}$ and, as presented in [17,19,38], it satisfies the *Kummer differential equation*

$$z \frac{\partial^2 M}{\partial z^2}(a, b, z) + (b - z) \frac{\partial M}{\partial z}(a, b, z) - aM(a, b, z) = 0. \tag{2.11}$$

As discussed in [17, 19, 38], for every $a, b, z \in \mathbb{C}$ such that $\Re(b) > \Re(a) > 0$, Kummer functions also admit the integral representation

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 \mathrm{e}^{zt} t^{a-1} (1 - t)^{b-a-1} \mathrm{d}t, \tag{2.12}$$

where Γ denotes the Gamma function. This integral representation has been exploited in [29] to characterize the spectrum of some DDEs.

Kummer functions satisfy some induction relations, often called *contiguous relations*, see for instance [38]. In particular, the following relations are of interest.

Lemma 2.4 ([38, p. 325])**.** *Let $a, b, z \in \mathbb{C}$ with $a \neq b$, $z \neq 0$, and $-b \notin \mathbb{N}$. The following relations hold:*

$$\begin{aligned} M(a, b + 1, z) =& \frac{-b\,(a + z)\; M(a, b, z) + a\, b\, M(a + 1, b, z)}{z\,(a - b)}, \\ M(a + 1, b + 1, z) =& -\frac{-b\, M(a + 1, b, z) + b\, M(a, b, z)}{z}. \end{aligned} \tag{2.13}$$

Kummer confluent hypergeometric functions have close links with *Whittaker functions*. For $k, l \in \mathbb{C}$ with $-2l \notin \mathbb{N}^*$, the *Whittaker function* $\mathcal{M}_{k,l}$ is the function defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_{k,l}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l} M(\tfrac{1}{2} + l - k, 1 + 2l, z), \tag{2.14}$$

(see, e.g., [38]). Note that, if $\frac{1}{2} + l$ is not an integer, the function $\mathcal{M}_{k,l}$ is a multi-valued complex function with branch point at $z = 0$. The nontrivial roots of $\mathcal{M}_{k,l}$ coincide with those of $M(\frac{1}{2} + l - k, 1 + 2l, \cdot)$ and $\mathcal{M}_{k,l}$ satisfies the *Whittaker differential equation*

$$\varphi''(z) = \left(\frac{1}{4} - \frac{k}{z} + \frac{l^2 - \frac{1}{4}}{z^2} \right) \varphi(z). \tag{2.15}$$

Since $\mathcal{M}_{k,l}$ is a nontrivial solution of the second-order linear differential equation (2.15), any nontrivial root of $\mathcal{M}_{k,l}$ is necessarily simple.

In [23], Hille studies the distribution of zeros of functions of a complex variable satisfying linear second-order homogeneous differential equations with variable coefficients, as is the case for the degenerate Whittaker function $\mathcal{M}_{k,l}$, which satisfies (2.15). Thanks to an integral transformation defined there and called *Green–Hille transformation*, and some further conditions on the behavior of the function, Hille showed how to discard regions in the complex plane in order to preclude complex roots.

The following result, which is proved in [11] using the Green–Hille transformation from [23], gives insights on the distribution of the nonasymptotic zeros of Kummer hypergeometric functions with real arguments a and b.

Proposition 2.5 ([11]). *Let a, $b \in \mathbb{R}$ be such that $b \geq 2$.*

(a) *If $b = 2a$, then all nontrivial roots z of $M(a, b, \cdot)$ are purely imaginary;*

(b) *If $b > 2a$ (resp., $b < 2a$), then all nontrivial roots z of $M(a, b, \cdot)$ satisfy $\Re(z) > 0$ (resp., $\Re(z) < 0$);*

(c) *If $b \neq 2a$, then all nontrivial roots z of $M(a, b, \cdot)$ satisfy*

$$(b - 2a)^2 \Im(z)^2 - (4a(b - a) - 2b)\, \Re(z)^2 > 0. \tag{2.16}$$

2.3 Background on Gröbner basis and elimination techniques

Gröbner basis provide a uniform computational technique for solving different problems that can be interpreted in terms of polynomial equations in an algorithmic or computational way. A wide range of problems related to the qualitative analysis of dynamical systems leads to systems of polynomial equations, see for instance [5, 9, 15, 43]. To fix better the idea, let $f_i \in \mathbb{K}[x_1, \ldots, x_n]$ with $\mathbb{K}$ an arbitrary field, such that

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_m(x_1, \ldots, x_n) = 0 \tag{2.17}$$

To solve this system, Gröbner basis computations can be used by considering the ideal $\langle f_1, \ldots, f_m \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$. In this section, we recall the basic facts about Gröbner basis, and refer the reader to [18] for details.

Definition 2.6. A *monomial* in $x_1, \ldots, x_n$ is a product of the form

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \tag{2.18}$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. The *total degree* of this monomial is the sum $\alpha_1 + \ldots + \alpha_n$.

A *monomial ordering* is a total order on monomials that is compatible with the product and such that every nonempty set has a smallest element for the order. The leading term of a polynomial is the greatest monomial appearing in this polynomial.

Definition 2.7. Let f be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$.

(a) We call a_α the *coefficient* of the monomial x^α;

(b) If $a_\alpha \neq 0$, then we call $a_\alpha\, x^\alpha$ a *term* of f;

(c) The *total degree* of f, denoted $deg(f)$, is the maximum $|\alpha|$ such that the coefficient a_α is nonzero.

A *Gröbner basis* of an ideal $\mathcal{I}$ for a given monomial ordering is a set G of generators of $\mathcal{I}$ such that the leading terms of G generate the ideal of leading terms of polynomials in $\mathcal{I}$. A polynomial is *reduced* with respect to the Gröbner basis G when its leading term is not a multiple of those of G. The basis is *reduced* if each element $g \in G$ is reduced with respect to $G \setminus \{g\}$. For a given monomial ordering, the reduced Gröbner basis of a given set of polynomials exists and is unique, and can be computed using one's favorite general computer algebra system, like `Maple`, Magma or Singular. Several efficient implementations of the Gröbner basis algorithm exist, here we use the FGb implementation of F_4 available in `Maple` [20]. The complexity of a Gröbner basis computation is well known to be generically exponential in the number of variables, and in the worst case doubly exponential in the number of variables. Moreover, a good choice of the monomial ordering reduces the computational cost.

The *graded reverse lexicographic order* or `grevlex` for short also denoted `tdeg` ordering is the most suited ordering for the computation of the (reduced) Gröbner basis. The monomials are first ordered by degree, and the order between two monomials of the same degree $x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $x_\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ is given by $x_\alpha \succ x_\beta$ when the last nonzero element of $(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ is negative. Thus, among the monomials of degree d, the order is

$$x_1^d \succ x_1^{d-1}x_2 \succ x_1^{d-2}x_2^2 \succ \cdots \succ x_2^d \succ x_1^{d-1}x_3 \succ x_1^{d-2}x_2x_3 \succ x_1^{d-2}x_3^2 \succ \cdots \succ x_n^d.$$

However, a Gröbner basis for the `grevlex` ordering may not be appropriate for the computation of the solutions of the system (2.17). The most suited ordering for this computation is the *lexicographical* ordering (or `lex` ordering for short). The monomials are ordered by comparing the exponents of the variables in lexicographical order. Thus, any monomial containing x_1 is greater than any monomial containing only variables $x_2, \ldots, x_n$.

Under some hypotheses (radical ideal with a finite number of solutions, and up to a linear change of coordinates), the Gröbner Basis of an ideal $\langle f_1, \ldots, f_m \rangle$ for the `lex` order $x_1 > \ldots > x_n$ has the shape

$$\{x_1 - g_1(x_n), x_2 - g_2(x_n), \ldots, x_{n-1} - g_{n-1}(x_{n-1}), g_n(x_n)\}\,, \tag{2.19}$$

where the g_i's are univariate polynomials. In this case, the computation of the solutions of the system follows easily. In the general case, the shape of the Gröbner basis for the `lex` ordering is more complicated, but it is equivalent to several triangular systems for which the computation of the solutions are straightforward.

An important point is that a Gröbner basis for the `lex` order is in general hard to compute directly. Practically, it is faster to compute first a Gröbner basis for the `grevlex` order, and then to make a change of ordering to the `lex` order.

Further, if one wants to compute all the polynomials in $\mathcal{I}$, that do not depend on the variables $x_i, \ldots, x_n$, i.e., $\mathcal{I} \cap \mathbb{K}[x_1, \ldots, x_{i-1}]$, an elimination procedure has to be carried out. Geometrically speaking, this elimination of indeterminates corresponds to the projection of the associated variety into $[x_1, \ldots, x_1]$. This projection can be characterized using a Gröbner basis of $\mathcal{I}$ for `lex` ordering often designating an *elimination ordering*.

The precise ordering we use to compute the Gröbner basis of the polynomial systems occurring in this paper are `grevlex` or `lexdeg` ordering.

Finally, we use repeatedly the *Radical Membership Theorem*:

Theorem 2.8 ([18]). *Let $\mathcal{I} = \langle f_1, \ldots, f_s \rangle$ be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$, then f belongs to $\sqrt{\mathcal{I}}$ if, and only if, $\langle f_1, \ldots, f_s, 1 - yf \rangle = \langle 1 \rangle = \mathbb{K}[x_1, \ldots, x_n, y]$.*

A finite set of polynomial equations $\langle f_1, \ldots, f_m \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$ is said to be inconsistent or unfeasible if, and only if, $\langle 1 \rangle$ is the corresponding reduced Gröbner basis.

In elimination methods, an efficient certified routine from the `RootFinding[Parametric]` package of computer algebra system `Maple` is called the `CellDecomposition`, it was introduced in `Maple 15`. This routine considers systems of parametric algebraic and semi-algebraic equations and decomposes the parameter space into cells in which the original system has a constant number of solutions, see for instance [25]. The command returns a data structure that can be used for plotting the regions of the parameter space for which the system has a given number of solutions, it can be used for extracting sample points in the parameter space for which the system has a given number of solutions, as it can be used for extracting boxes in the parameter space in which the system has a given number of solutions. The input parametric polynomial system must satisfy the following properties:

(a) The number of equations is equal to or greater than the number of indeterminates (parameters).

(b) At most finitely many complex solutions exist for almost all complex parameter values (that is, the system is generically zero-dimensional).

(c) For almost all complex parameter values, there are no solutions of multiplicity greater than 1 (that is, the system is generically radical); in particular, the input equations are square-free.

3 Multiplicity-Induced-Dominancy approach

Several methods and techniques has been used to prove the validity of the MID property such as, in the second-order case, the principle argument under the formalism of Stépán-Hassard in [12] and the link with the distribution of hypergeometric functions' zeros in the GMID property in arbitrary order case in [10]. The next section provides a comprehensive algorithm allowing to parametrically characterize both GMID and IMID properties.

3.1 MID methodology

The MID property consists of the conditions under which a given multiple root of a quasipolynomial function is necessarily dominant. Notice that in the generic quasipolynomial function case, the real root of maximal multiplicity is necessarily the dominant (GMID), see for instance [7, 8] for first and second order time-delay equation. However, multiple roots with intermediate admissible multiplicities may be dominant or not. Thanks to this property, an ensued control strategy is proposed in [4, 12], which consists in assigning a root with an admissible multiplicity once appropriate conditions guaranteeing its dominancy have been established.

The proof of the MID property consists of five steps. First, we establish conditions on the parameters of the system guaranteeing the existence of a multiple root. Second, an affine change of variable of the characteristic equation is performed in order to reduce the said quasipolynomial to a normalized form; the desired multiple root becomes 0 and the delay 1. Next, under the latter normalization, the characteristic equation may be easily factorized in terms of an integral expression. Hence, we derive a bound on the imaginary

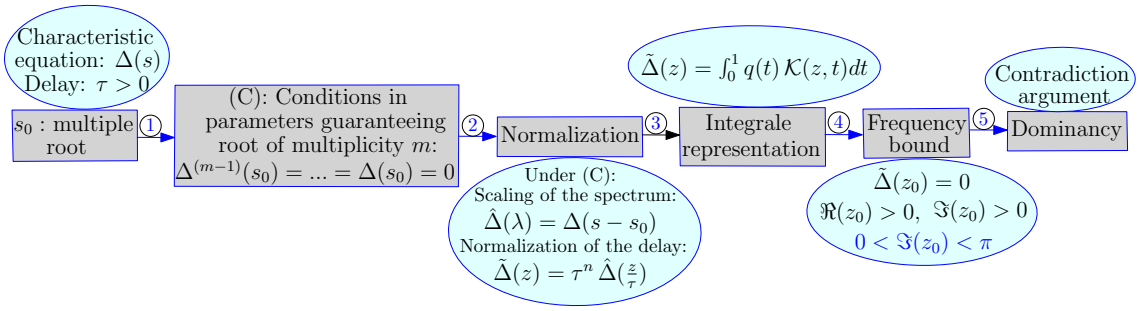

Diagram illustrating the proof methodology of the MID property for time-delay differential equations.

part of the roots of the normalized quasipolynomial in the complex right half-plane. Lastly, a certification of the dominance of the multiple root is demonstrated.

The MID methodology is fully detailed in [7] for the case of second order time-delay equation.

3.2 Frequency bound technique

The fourth step in the MID methodology is to derive an appropriate bound on the imaginary part of roots of the normalized quasipolynomial in the complex right half-plane. In the following, Algorithm 1 (see [7]) is a pseudo-code listing the instructions to be followed to target a suitable frequency bound.

Algorithm 1: Estimation of a frequency bound for time-delay differential equations with a single delay

Input: $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)\,\mathrm{e}^{-z}$; `// Normalized quasipolynomial`

Input: `maxOrd`; `// Maximal order`

`// Initialization`

`ord` $= 0$; `// ord: order of truncation of the Taylor expansion of` e^{2x};

`dominance` $=$ `false`;

while *(not* `dominance`*) and (*`ord` $\leq$ `maxOrd`*)* **do**

  Set $\mathcal{F}(x,\omega) = |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2 T_{\mathtt{ord}}(x)$;

  `//` $T_{\mathtt{ord}}(x)$: `Taylor expansion of` e^{2x} `of order =` `ord`

  Set $H(x,\Omega) = \mathcal{F}(x,\sqrt{\Omega})$; `//` H `is a polynomial`

  Set $\Omega_k(x)$ as the k-th real root of $H(x,\cdot)$;

  if $\sup_{x\geq 0} \max_k \Omega_k(x) \leq \pi^2$ **then**

    `dominance` $=$ `true`;

  `ord` $=$ `ord` $+ 1$;

Output: Frequency bound: If `dominance` is `true`, then $|\omega| \leq \pi$ for every root of $\tilde{\Delta}$ with positive real part;

3.3 A toy model

In this section, we present a case study (see [12, 29]). It corresponds to the model of phenomena in the bio-sciences describing the dynamics of a vector-borne disease.

Consider a model based on a simple scalar delay differential equation with a positive single delay τ. In its linearized version, the infected host population $x(t)$ is governed by

$$\dot{x}(t) + a_0\, x(t) + a_1\, x(t-\tau) = 0, \tag{3.1}$$

where $a_1 > 0$ is called the contact rate; it represents the contact number between infected and uninfected populations. Assume that the infection of the host recovery proceeds exponentially at a rate of $-a_0 > 0$. The characteristic equation of (3.1) is the quasipolynomial function of degree 2, defined by

$$\Delta(s) = s + a_0 + a_1\, \mathrm{e}^{-\tau\, s}. \tag{3.2}$$

The characteristic equation (3.2) admits a real root s_0 of maximal multiplicity 2 if, and only if,

$$s_0 = -\frac{a_0\,\tau + 1}{\tau}, \quad a_1 = \frac{\mathrm{e}^{s_0\,\tau}}{\tau}. \tag{3.3}$$

It was emphasized that s_0 is the rightmost root, and that if $s_0 < 0$ then the zero solution of system (3.2) is asymptotically stable.

In order to prove that s_0 is a dominant root of (3.2), we factorize the characteristic equation Δ under an integral representation as:

$$\Delta(s) = (s - s_0)\, \left(1 - \int_0^1 \mathrm{e}^{-\tau\,(s-s_0)\,t}\right) \mathrm{d}\, t \tag{3.4}$$

which cannot be satisfied for any spectral value s with $\Re(s) > s_0$.

As a matter of fact, if $s_1 = x + i\,\omega$ such that $s_1 \neq 0$ is a root of the factorized quasipolynomial (3.4), then, s_1 is a root of its second factor, i.e.,

$$1 - \int_0^1 \mathrm{e}^{-\tau\,(s_1-s_0)\,t} = 0.$$

Hence, we obtain

$$1 = \int_0^1 \mathrm{e}^{-\tau\,(x-s_0)\,t}. \tag{3.5}$$

But, $\mathrm{e}^{-\tau\,(x-s_0)\,t} < 1$ for $x - s_0 > 0$ and $t \in (0, 1)$ which leads to the dominancy of s_0.

3.4 Parametric MID property in second-order systems

The control design method resulting from the MID property, which consists in forcing a given quasipolynomial to have a root of a prescribed multiplicity, allows under appropriate conditions to characterize the rightmost root. Indeed, this multiplicity constraint defines a manifold in the parameter space enabling the tuning of the gains b_k when the delay τ is left free and guaranteeing the exponential stability of the closed-loop system solution. The next theorem, which is based on the MID property, gives the explicit conditions on the parameters' values guaranteeing a targeted multiplicity. Recall that the multiplicity of a given root of the generic quasipolynomial (2.1) is bounded by its degree; that is, 4 is the bound of multiplicity. However, forcing such a multiplicity must not constrain the physical model's parameters.

Theorem 3.1 (**GMID**). *Let* Δ *be the generic quasipolynomial function in* (2.1). *The following assertions hold:*

(a) *The multiplicity of a given spectral value* s_0 *of the quasipolynomial function* Δ *is bounded by* 4.

(b) *The quasipolynomial function* Δ *admits a real spectral value of maximal multiplicity* 4 *if, and only if, the following relations are satisfied*

$$\begin{cases} a_0 = & \frac{{s_0}^2\tau^2+4\,\tau\,s_0+6}{\tau^2}, \\ a_1 = & -2\,\frac{\tau\,s_0+2}{\tau}, \\ b_0 = & 2\,\frac{\tau\,s_0-3}{\tau^2 \mathrm{e}^{-\tau\,s_0}}, \\ b_1 = & -2\,\frac{1}{\tau\,\mathrm{e}^{-\tau\,s_0}}. \end{cases} \tag{3.6}$$

In addition, if relations (3.6) *are satisfied, then* s_0 *is necessarily a dominant root of* Δ.

Proof. First, we write the generic quasipolynomial function (2.1) under the form

$$\Delta(s) = P_0(s) + P_\tau(s)\,\mathrm{e}^{-\tau\,s}, \tag{3.7}$$

where

$$\begin{cases} P_0(s) & = s^2 + a_1\,s + a_0, \\ P_\tau(s) & = b_1\,s + b_0. \end{cases} \tag{3.8}$$

Normalization: Performing the translation and scaling of the spectrum by the following change of variables $\hat{\Delta}(z) = \tau\,\Delta(\frac{z}{\tau} + s_0)$ for $z \in \mathbb{C}$, we get the following normalized characteristic equation

$$\hat{\Delta}(z) = z^2 + \alpha_1\,z + \alpha_0 + (\beta_1 z + \beta_0)\,\mathrm{e}^{-z}$$

with relations (3.6) normalized as follows

$$\begin{cases} \alpha_1 = & (a_1 + 2\,s_0)\,\tau, \\ \alpha_0 = & (a_1 s_0 + {s_0}^2 + a_0)\,\tau^2, \\ \beta_0 = & (b_1 s_0 + b_0)\,\tau^2\,\mathrm{e}^{-\tau\,s_0}, \\ \beta_1 = & b_1\,\tau\,\mathrm{e}^{-\tau\,s_0}. \end{cases} \tag{3.9}$$

Integral representation: The real root s_0 is a root of multiplicity 4 of Δ if, and only if, 0 is a root of multiplicity 4 of $\hat{\Delta}$. As a matter of fact, since $\hat{\Delta}$ is a quasipolynomial of degree 4, zero is a root of multiplicity 4 of $\hat{\Delta}$ if, and only if, $\hat{\Delta}(0) = \hat{\Delta}'(0) = \hat{\Delta}''(0) = \hat{\Delta}'''(0) = 0$. These identities yield a linear system whose unique solution is $(\alpha_1, \alpha_0, \beta_1, \beta_0) = (-4, 6, -2, -6)$. From relations (3.9), one concludes that s_0 is a root of multiplicity 4 of Δ if, and only if, relations (3.6) hold. Moreover, under the latter conditions, the quasipolynomial Δ reduces to

$$\hat{\Delta}(z) = z^2 - 4\,z + 6 + (-2\,z - 6)\,\mathrm{e}^{-z} \tag{3.10}$$

Following [10] the normalized quasipolynomial $\hat{\Delta}$ can be written in terms of Kummer hypergeometric functions as follows:

$$\hat{\Delta}(z) = \frac{1}{12}\,z^4\,M(2, 5, -z).$$

Also, the quasipolynomial $\hat{\Delta}$ admits the following Fredholm integral representation

$$\hat{\Delta}(z) = \int_0^1 q(t)\,\mathcal{K}(z,t)\,\mathrm{d}t$$

where

$$q(t) = t\ (t-1)^2\,, \quad \text{and} \quad \mathcal{K}(z,t) = z^4 \mathrm{e}^{-t\,z},$$

which is easily verified via an integration by parts. Note that in our approach, the polynomial q should keep a constant sign for $t \in (0,1)$, which is satisfied here.

Frequency bound: We write the quasipolynomial $\hat{\Delta}$ under the form

$$\hat{\Delta}(z) = \hat{P}_0(z) + \hat{P}_1(z)\,\mathrm{e}^{-z}, \tag{3.11}$$

with

$$\begin{cases} \hat{P}_0(z) = & z^2 - 4\,z + 6, \\ \hat{P}_1(z) = & -2\,z - 6. \end{cases} \tag{3.12}$$

and assume that $z = x + i\omega \in \mathbb{R}_+ + i\mathbb{R}_+$ is a root of $\hat{\Delta}$, so that $\hat{\Delta}(z) = 0$ if, and only if,

$$|\hat{P}_0(x+i\omega)|^2\,\mathrm{e}^{2\,x} = |\hat{P}_1(x+i\omega)|^2.$$

Considering a truncation of order 1 of the exponential term $\mathrm{e}^{2\,x}$, the latter is lower bounded by $1 + 2x$. Next, define

$$F(x,\omega) = |\hat{P}_1(x+i\omega)|^2 - (1+2x)\,|\hat{P}_0(x+i\omega)|^2$$

where $F > 0$ for any $x > 0$. The zeros of F are characterized by the first order polynomial

$$G(\Omega, x) = -\,(1+2\,x)\,\Omega^2 - 2\,x^2\,(2\,x-7)\,\Omega - 2\,x^5 + 15\,x^4 - 48\,x^3 + 72\,x^2,$$

where $\Omega = \omega^2$. The polynomial function G admits for $x \in (0, \frac{3}{2} + \frac{3\sqrt{2}}{2})$ two real roots:

$$\Omega^{\pm}(x) = \frac{x}{1+2\,x}\left(-2\,x^2 \pm 2\,\sqrt{-8\,x^2 + 24\,x + 18} + 7\,x\right) \tag{3.13}$$

where Ω^+ denotes the greater solution at x which reaches a maximum value at $x^* \approx 1,93$. As a result, Ω^+ is bounded by $\Omega^* \approx 7,07 < \pi^2$. Thus, one obtains the desired frequency bound, $0 < \omega < \pi$.

Dominancy: The proof of the dominance is based on a contradiction. To do so, assume that there exists $z = x + i\omega \in \mathbb{R}_+ + i\mathbb{R}_+$ root of $\hat{\Delta}$. Using the fact that z is a non-zero root of $\hat{\Delta}$, one may infer from the Fredholm integral representation, by taking the imaginary part, that

$$\int_0^1 t\ (t-1)^2\ \sin(t\,\omega)\,\mathrm{e}^{-t\,x}\,\mathrm{d}t = 0.$$

Since $\omega < \pi$ from the previous step (Frequency bound), the function $t \mapsto t\ (t-1)^2\ \sin(t\,\omega)$ is strictly positive in $(0,1)$, which contradicts the above equality as required to end the proof.

□

Theorem 3.2 (**IMID**: codimension 3)**.** *Let Δ be the generic quasipolynomial function in* (2.1)*. The following assertions hold.*

(a) *If the parameters a_0, a_1 and τ are left free, then s_0 is a real root of multiplicity 3 of* (2.1) *if, and only if, s_0 is a root of the* elimination-produced polynomial

$$\mathbb{P}(s) = \tau^2 s^2 + (a_1\tau + 4)\ \tau\,s + \tau^2 a_0 + 2\,\tau\,a_1 + 2. \tag{3.14}$$

(b) *The root* s_0 *of* (2.1) *has multiplicity* 3 *if, and only if,* s_0 *is real and the system parameters satisfy:*

$$\begin{cases} b_0 = \dfrac{\left(\tau^2 a_1 s_0 + 2\,\tau^2 a_0 + 6\,\tau\,a_1 + 10\,\tau\,s_0 + 6\right)\mathrm{e}^{\tau\,s_0}}{\tau^2}, \\ b_1 = \dfrac{\left(\tau\,a_1 + 2\,\tau\,s_0 + 2\right)\mathrm{e}^{\tau\,s_0}}{\tau}. \end{cases} \tag{3.15}$$

(c) *In addition, if the following conditions are satisfied:*

(i) $-\frac{2\,(\tau\,s_0+1)}{\tau} < a_1 < -2\,s_0,$

(ii) $\frac{-8-(a_1+2\,s_0)^2\tau^2+(-8\,a_1-16\,s_0)\tau}{4\,\tau^2} < a_0 < \frac{-2+(-2\,a_1-4\,s_0)\tau}{\tau^2} \quad (a_1 \le 0)\vee\left(0 < a_1 \wedge s_0 < -\frac{a_1}{2}\right),$

then the assigned root s_0 *is necessarily a dominant root of* Δ.

The proof of item (c) in Theorem 3.2 requires the following propositions dealing with algebraic properties of polynomials.

Proposition 3.3 (Hurwitz and Real-rooted elimination-produced-polynomial)**.** *Let* $\mathbb{P}$ *be the elimination-produced polynomial defined in* (3.14)*, then* $\mathbb{P}$ *admits two negative roots if, and only if,*

(a) $a_1 < -2\,s_0,$

(b) $\frac{-8-(a_1+2\,s_0)^2\tau^2+(-8\,a_1-16\,s_0)\tau}{4\,\tau^2} < a_0 < \frac{-2+(-2\,a_1-4\,s_0)\tau}{\tau^2} \quad (a_1 \le 0)\vee\left(0 < a_1 \wedge \lambda_0 < -\frac{a_1}{2}\right).$

Proof. Consider the following new parametrization

$$\sigma = (a_1 + 2\,s_0)\,\tau, \quad \delta = a_0\,\tau^2, \quad z = s_0\,\tau. \tag{3.16}$$

Under the above reparametrization, the elimination-produced polynomial $\mathbb{P}$ in (3.14) is reduced to

$$\tilde{\mathbb{P}}(z) = -z^2 + (z+2)\,\sigma + \delta + 2. \tag{3.17}$$

Using the `Maple` routine `CellDecomposition` of the `RootFinding[Parametric]` package, considering that $z < 0$, we compute a set of polynomials whose zeros define a discriminant variety of the parametric equation $\tilde{\mathbb{P}}(z) = 0$ as well as a cylindrical algebraic decomposition of the complementary of this discriminant variety, which yields six cells C_i, for $i = 1, 2, ..., 6$ (see Figure 3.1). For each cell, we have a sampling point strictly in the interior of the cell. Only one cell is of interest that is C_2 as shown in figure 3.1. A description of C_2 in terms of real roots is to be interpreted as follows: a point $[\delta, \sigma]$ in the parameter space belongs to the cell C_2 if, and only if,

$$\sigma < 0, \quad \text{and} \quad -\frac{\sigma^2}{4} - 2\,\sigma - 2 < \delta < -2\,\sigma - 2, \tag{3.18}$$

which concludes the announced result. □

In the following, we proceed with the normalization and integral representation of the characteristic equation Δ.

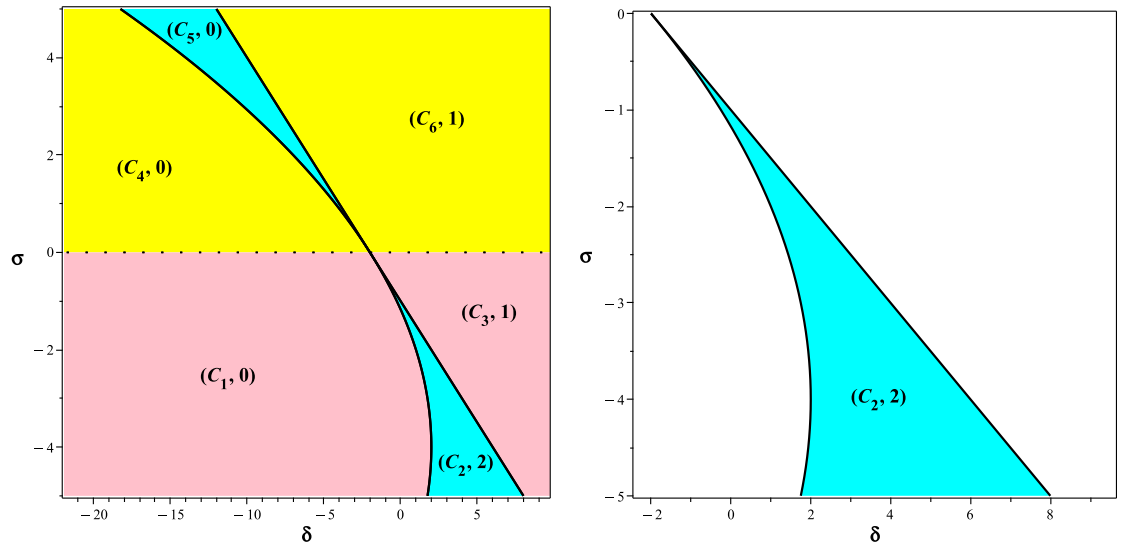

Figure 3.1: (Left): A chart of the number of real zeros in the parameter space (δ, σ). (Right): The cell C_2 where there exist two negative roots for the elimination-produced-polynomial $\tilde{\mathbb{P}}$ ($\tilde{\mathbb{P}}$ is real-rooted and has negative roots). The notation (C_i, n) for $i = 1, \cdots, 6$, stands for the i^{th} cell with n the corresponding number of zeros in the cell satisfying some algebraic equations.

Proposition 3.4 (Normalization and Integral representation of the characteristic function)**.** *Let Δ be the quasipolynomial function in 2.1, then the normalization of Δ under conditions* (3.15) *is given by*

$$\tilde{\Delta}(z) = ((\sigma + 2)\, z + 2\,\sigma + 2)\, \mathrm{e}^{-z} + z^2 + \sigma\, z - 2\,\sigma - 2, \tag{3.19}$$

where the parameter σ depends on the original parameters a_1, s_0 and τ as defined in (3.16)*. In addition, the quasipolynomial function $\tilde{\Delta}$ may be represented under integral form as follows:*

$$\tilde{\Delta}(z) = \int_0^1 q_\sigma(t)\, \mathcal{K}(z,t)\, \mathrm{d}t \tag{3.20}$$

where

$$q_\sigma(t) = (-\sigma - 1)\, t^2 + \sigma\, t + 1, \quad \textit{and} \quad \mathcal{K}(z,t) = z^3 \mathrm{e}^{-t\, z}. \tag{3.21}$$

In our approach, the sign constancy of the polynomial q_σ defined previously for $t \in (0,1)$ is necessary. It is the purpose of the next proposition.

Proposition 3.5 (Constancy sign of q_σ)**.** *Let q_σ be the polynomial with respect to t defined in* (3.21)*. Then, q_σ keeps a constant sign for $t \in (0,1)$ if, and only if,*

$$a_1 > -2 \left(s_0 + \frac{1}{\tau} \right) \tag{3.22}$$

Proof. Note that q_σ admits two real roots:

$$t_1 = 1, \quad \text{and} \quad t_2 = -\left(\sigma + 1\right)^{-1}. \tag{3.23}$$

Therefore, q_σ is of constant sign for $t \in (0,1)$, if the real root t_2 satisfies either $t_2 \leq 0$ or $t_2 > 1$. We conclude that q_σ is of constant sign for $t \in (0,1)$ if, and only if ,

$$\sigma > -2, \tag{3.24}$$

which completes the proof. □

By the following proposition we characterize the region in the parameters' space where the elimination-produced-polynomial $\mathbb{P}$ is Hurwitz and real-rooted, and the integrand q_σ is of constant sign for $t \in (0,1)$.

Proposition 3.6 (Admissible region for a frequency bound)**.** *Let $\mathbb{P}$ be the elimination-produced-polynomial $\mathbb{P}$ defined in* (3.14)*, and q_σ defined in* (3.21)*. Then, $\mathbb{P}$ is Hurwitz, real-rooted and q_σ is of constant sign for $t \in (0,1)$ if, and only, if*

(a) $-\frac{2\,(\tau\, s_0+1)}{\tau} < a_1 < -2\, s_0,$

(b) $\frac{-8-(a_1+2\, s_0)^2\tau^2+(-8\, a_1-16\, s_0)\tau}{4\,\tau^2} < a_0 < \frac{-2+(-2\, a_1-4\, s_0)\tau}{\tau^2} \quad (a_1 \le 0)\vee(0 < a_1 \wedge s_0 < -\frac{a_1}{2}).$

Proof. In a similar way as for the proof of Proposition 3.3, we use again the `Maple` routine `CellDecomposition` with the addition, this time around, of the conditions of the constancy of the sign of q_σ in (3.24) to the vanishing of the elimination-produced-polynomial $\tilde{\mathbb{P}}(z) = 0$, as well as the negativity of the assigned root $z < 0$. Hence, we obtain nine cells only one of which is of interest, that is, C_5, as shown in figure 3.2. By exploiting its algebraic characterization, we infer the announced result. □

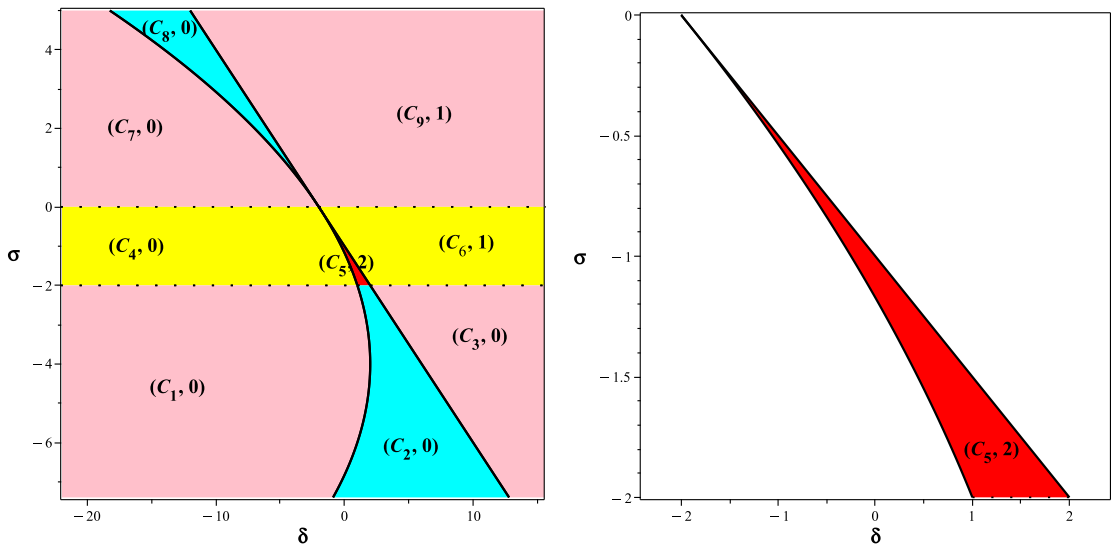

Figure 3.2: (Left): A chart of the number of real zeros in the parameter space (δ, σ) where the constancy sign of q_σ is satisfied for $t \in (0,1)$. (Right): The cell C_5 where there exist two negative roots for the elimination-produced-polynomial $\mathbb{P}$ (it is real rooted and has negative roots). The notation (C_i, n) for $i = 1, \cdots, 9$, stands for the i^{th} cell with n the corresponding number of zeros in the cell satisfying some semi-algebraic equations.

After characterizing the regions where q_σ is of constant sign for $t \in (0,1)$ and $\mathbb{P}$ is Hurwitz and real-rooted, we are able to establish a frequency bound on roots of $\tilde{\Delta}$.

Proposition 3.7 (Frequency bound)**.** *Let $\tilde{\Delta}$ be the quasipolynomial function given in* (3.19)*. If $\tilde{\Delta}$ has a root $s \in \mathbb{R}_+ + i\mathbb{R}_+$ and $\sigma \in (-2, 0)$, then $0 < \Im(s) < \pi$.*

Proof. Let $z = x + i\omega \in \mathbb{R}_+ + i\,\mathbb{R}_+$ be a root of

$$\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_1(z)\,\mathrm{e}^{-z} \tag{3.25}$$

with

$$\tilde{P}_1(z) = (\sigma + 2)\, z + 2\,\sigma + 2, \tag{3.26}$$

$$\tilde{P}_0(z) = z^2 + \sigma\, z - 2\,\sigma - 2, \tag{3.27}$$

as defined in (3.19). The complex number z satisfies the following equality

$$|\tilde{P}_0(x + i\omega)|^2 \mathrm{e}^{2\,x} = |\tilde{P}_1(x + i\omega)|^2. \tag{3.28}$$

Since $\mathrm{e}^{2\,x} > 1$, the function

$$F_\sigma(x, \omega) = |\tilde{P}_1(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 \tag{3.29}$$

satisfies $F_\sigma(x, \omega) > 0$. Moreover, the zeros of F_σ can be characterized by a polynomial of degree 2 in $\Omega = \omega^2$:

$$G_\sigma(\Omega, x) = -\Omega^2 + \left(-2\,\sigma\, x - 2\, x^2\right) \Omega - x^4 - 2\,\sigma\, x^3 + (8\,\sigma + 8)\, x^2 + 8\, (\sigma + 1)^2\, x. \tag{3.30}$$

In our approach, one needs to guarantee the positivity of G_σ, i.e., one has to investigate conditions on the sign of the discriminant of G_σ which is defined by the following second degree polynomial in x

$$D_\sigma(x) = 4\, \left(\sigma^2 + 8\,\sigma + 8\right) x^2 + 32\, (\sigma + 1)^2\, x. \tag{3.31}$$

The latter discriminant satisfies the following.

- If $\sigma \in (-4 + 2\sqrt{2}, 0)$, then $D_\sigma > 0$ for $x \in (x_0, \infty)$
- If $\sigma \in (-2, -4 - 2\sqrt{2})$, then $D_\sigma > 0$ for $x \in (0, x_0)$

where

$$x_0 = -\frac{8\, (\sigma + 1)^2}{\sigma^2 + 8\,\sigma + 8}. \tag{3.32}$$

In both cases above, G_σ admits the following two real roots

$$\Omega_\sigma^{\pm}(x) = -x^2 - \sigma\, x \pm \sqrt{(\sigma^2 + 8\,\sigma + 8)\, x^2 + (\sigma^2 + 16\,\sigma + 8)\, x} \tag{3.33}$$

where Ω_σ^+ denotes the greater solution which is upper bounded with respect to σ by

$$\Omega^+(x) = -x^2 + \sqrt{8\, x^2 + 8\, x}. \tag{3.34}$$

The above function in (3.34) reaches a maximum value at $x^* \approx 1.46$. As a result, Ω_σ^+ is bounded by $\Omega^+(x^*) \approx 3.23 < \pi^2$. Thus, one obtains the desired frequency bound $\omega < \pi$. Figure 3.3 shows how the parameter expression of Ω_σ^+ is bounded by $\Omega^+(x^*)$ for $\sigma \in (-2, 0)$.

□

Proof. (Proof of Theorem 3.2) First, equation $\Delta(s) = 0$, yields the elimination of the exponential term as a rational function in s:

$$\mathrm{e}^{-\tau\, s} = -\frac{P_0(s)}{P_\tau(s)}. \tag{3.35}$$

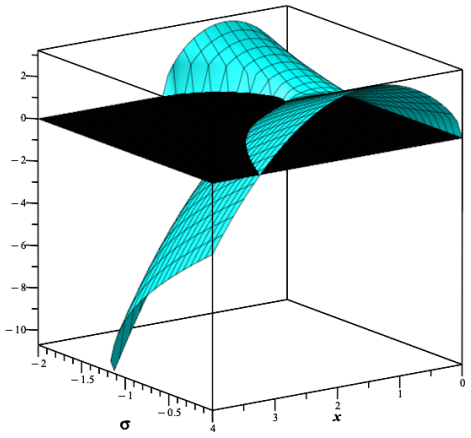

Figure 3.3: A three-dimensional plot corresponding to Ω_σ^+, showing its maximum is reached at the boundary of $\sigma \in (-2, 0)$.

Next, to investigate potential roots with algebraic multiplicity 3, one substitutes the exponential term as obtained in (3.35) in the ideal generated by the first and second derivatives of Δ, that is, $\mathcal{I}_2 =< \partial_s\Delta,\, \partial_s^2\Delta >$. This leads to the investigation of the following variety of two algebraic equations in 6 unknowns b_0, b_1, a_0, a_1, s, τ :

$$\begin{cases} \left(-s^2\tau + (-\tau\, a_1 + 2)\, s - \tau\, a_0 + a_1\right) b_0 \\ + \left(-s^3\tau + (-\tau\, a_1 + 3)\, s^2 + (-\tau\, a_0 + 2\, a_1)\, s + a_0\right) b_1 = 0, \\ \left(s^2\tau^2 + s\tau^2 a_1 + \tau^2 a_0 + 2\right) b_0 \\ + \left(s^3\tau^2 + \left(\tau^2 a_1 - 2\,\tau\right) s^2 + \left(\tau^2 a_0 - 2\,\tau\, a_1 + 2\right) s - 2\,\tau\, a_0\right) b_1 = 0. \end{cases} \tag{3.36}$$

The above system is a linear system in the unknowns b_0 and b_1. Using standard elimination techniques, one obtains a solution that can be substituted in (3.35) which yields the explicit values of the gain b_k $(k = 0, 1)$ allowing to tune the parameters as provided in (3.15).

Now, assume that there exists $s^* \in \mathbb{C} - \mathbb{R}$ such that $\mathbb{P}(s^*) = 0$ and the coefficients $b_k(s^*,\, a_1,\, \tau) \in \mathbb{R}$ $(k = 0, 1)$. So that, one substitutes $s^* = x + i\,\omega$ with $x \neq 0$ and $\omega \neq 0$ in the expressions of b_k $(k = 0, 1)$ and defines an ideal $\mathcal{I}_\mathcal{R}$ (where the index $\mathcal{R}$ stands for the realness of the gains b_k).

As suggested in the previous Section 2.3, one computes a Gröbner basis with respect to the elimination order `lex`; eliminating the unknowns a_1 and τ. Hence, a basis of 3 elements $G = \langle G_1, G_2, G_3 \rangle$ is incurred where

$$\begin{aligned} G_1 &= \omega^2, \\ G_2 &= 4\,x^2 + 4\,x a_1 + 3\,{a_1}^2 - 8\,a_0, \\ G_3 &= (-{a_1}^2 + 4\,a_0)\tau + 4\,x + 2\,a_1. \end{aligned}$$

The first element of the Gröbner basis G indicates that a solution of the multivariate system exists only if $\omega = 0$.

In order to prove item (c), Propositions 3.6 gives the conditions on the systems' parameters a_0 and a_1. To conclude the dominance, we proceed by contradiction. To do so, assume that there exist $z = x + i\,\omega \in \mathbb{R}_+ + i\,\mathbb{R}_+$ root of $\tilde{\Delta}$, then using the fact that z is a non-zero root of $\tilde{\Delta}$, we infer from the integral representation of $\tilde{\Delta}$, by taking the imaginary part, that

$$\int_0^1 \left[(-\sigma - 1)\,t^2 + \sigma\,t + 1\right] \sin(t\,\omega)\,\mathrm{e}^{-t\,x}\,\mathrm{d}t = 0. \tag{3.37}$$

Since $\omega < \pi$, following Proposition 3.7 the function $t \mapsto \left((-\sigma - 1)\,t^2 + \sigma\,t + 1\right) \sin(t\,\omega)$ is strictly positive in $(0, 1)$, which contradicts the above equality as required to end the proof. □

Remark 3.8. From a control viewpoint, the controllers' gains have to be real. What we care about is related to the elimination-produced polynomial. In fact, the elimination-produced polynomial should have at least one real-root but since it is of degree 2 then only one case is of interest; the elimination-produced polynomial with two real roots. More precisely, in the case of real-rooted elimination-produced polynomial, the corresponding roots need to be negative to be admissible assignable roots for the initial quasipolynomial Δ.

Remark 3.9. Note that the quasipolynomial Δ can be further written as a combination of two Kummer functions as

$$\tilde{\Delta}(z) = z^3 \left[\frac{1}{3}\,(-1 - \sigma)\,M\,(1,\,4,\,-z) + \frac{1}{2}\left(\sigma + \frac{1}{2}\right) M\,(1,\,3,\,-z)\right]. \tag{3.38}$$

Remark 3.10. Notice that an alternative proof relying on of the principle argument and using the Stépán-Hassard approach can be found in [12]. It is of important note that s_+ of Theorem 4.2 [12] corresponds to s_0 in Theorem 3.2.

4 Damping oscillations

In this section, we consider the oscillation damping of the classical oscillator. To this aim, consider the following control system

$$\ddot{x}(t) + 2\,\omega\,\xi\,\dot{x}(t) + \omega^2 x(t) = u(t) \tag{4.1}$$

where $x(t)$ is a real valued function, $0 < \xi < 1$ and $\omega \in \mathbb{R}_+$ are respectively the damping factor and the natural frequency of the oscillator and $u(t)$ is the delayed output of the system. Under no control, the characteristic polynomial of this equation is:

$$\Delta_0(s) = s^2 + 2\xi\,\omega\,s + \omega^2, \tag{4.2}$$

and the corresponding spectral abscissa is given by

$$\rho_0 = -\xi\,\omega.$$

In particular, $\rho_0 < 0$ and the system is exponentially stable. Choosing now a control:

$$u(t) = -\alpha_1\,\dot{x}(t - \tau) - \alpha_0\,x(t - \tau),$$

where $\alpha_0, \alpha_1 \in \mathbb{R}$, then, the closed-loop equation yields:

$$\Delta(s) = s^2 + 2\xi\,\omega\,s + \omega^2 + (\alpha_1\,s + \alpha_0)\,\mathrm{e}^{-\tau\,s}.$$

In order to place the characteristic roots, one may appropriately choose the coefficients α_0 and α_1 which allow to obtain arbitrary values for the corresponding spectral abscissa, and hence arbitrary exponential decay rates for the solutions. In particular, assigning a negative dominant spectral value enables the oscillation quenching.

Theorem 3.1 ensures that the quasipolynomial function Δ admits a real spectral value of maximal multiplicity 4 if, and only if, the following relations are satisfied

$$\begin{cases} \omega = & \frac{\sqrt{s_0^2\tau^2+4\tau s_0+6}}{\tau}, \\ \xi = & -\frac{\tau s_0+2}{\sqrt{s_0^2\tau^2+4\tau s_0+6}}, \\ b_0 = & \frac{2\tau s_0-6}{\tau^2 \mathrm{e}^{-\tau s_0}}, \\ b_1 = & -\frac{2}{\tau\, \mathrm{e}^{-\tau s_0}}, \end{cases} \tag{4.3}$$

and under which s_0 is necessarily the dominant root of Δ. Interestingly, solving the first two equalities in (4.3), yields the precise value of the assigned root s_0 as well as the delay which is seen as a control parameter. Indeed, one obtains:

$$\begin{cases} s_0 = & -\frac{\left(\sqrt{2}\,\xi+2\sqrt{-\xi^2+1}\right)\sqrt{2}\,\omega}{2}, \\ \tau = & \frac{\sqrt{2}}{\sqrt{-\xi^2+1}\,\omega}, \end{cases} \tag{4.4}$$

Obviously, the assigned root s_0 given in (4.4) not only eliminates the closed-loop system oscillations but also improves the corresponding decay rate.

As precised in [10], the GMID does not allow any degree of freedom in assigning s_0. In order to allow for some additional freedom when assigning s_0, which is important from robustness perspectives, one can relax such a constraint by forcing the root s_0 to have a multiplicity lower than the maximal.

Theorem 3.2 also asserts that s_0 is a real root of multiplicity 3 of (2.1) if, and only if, s_0 is a root of the elimination-produced polynomial

$$\mathbb{P}(s) = \tau^2 s^2 + (2\omega\tau\xi + 4)\,\tau s + \omega^2\tau^2 + 4\omega\tau\xi + 2. \tag{4.5}$$

The root s_0 of (2.1) has multiplicity 3 if, and only if, the control gains satisfy:

$$\begin{cases} b_0 = \frac{2\,\mathrm{e}^{\tau s_0}\left(3+\omega\left(\xi s_0+\omega\right)\tau^2+\left(6\xi\omega+5s_0\right)\tau\right)}{\tau^2}, \\ b_1 = \frac{2\,\mathrm{e}^{\tau s_0}\left(1+\left(\xi\omega+s_0\right)\tau\right)}{\tau}. \end{cases} \tag{4.6}$$

From Theorem (3.2), an effective admissible region for the frequency bound is described by the following conditions on the two parameters ω and ξ

(a) $-\frac{(\tau\, s_0+1)}{\tau\,\omega} < \xi < -\frac{\omega}{s_0}$, with $s_0 < -\frac{1}{\tau} \vee s_0 < -\omega$

(b) $\sqrt{\frac{-2-(\xi\omega+s_0)^2\tau^2-(4\xi\omega+4s_0)\tau}{\tau^2}} < \omega < \sqrt{\frac{-2(2\omega\tau\xi+2\tau s_0+1)}{\tau^2}}$, with

$$\frac{-2-\sqrt{2}-\omega\tau\xi}{\tau} < s_0 < \frac{-2+\sqrt{2}-\omega\tau\xi}{\tau}.$$

Remark 4.1. The obtained constraints on the root s_0 represent a valuable information on the assignment region.

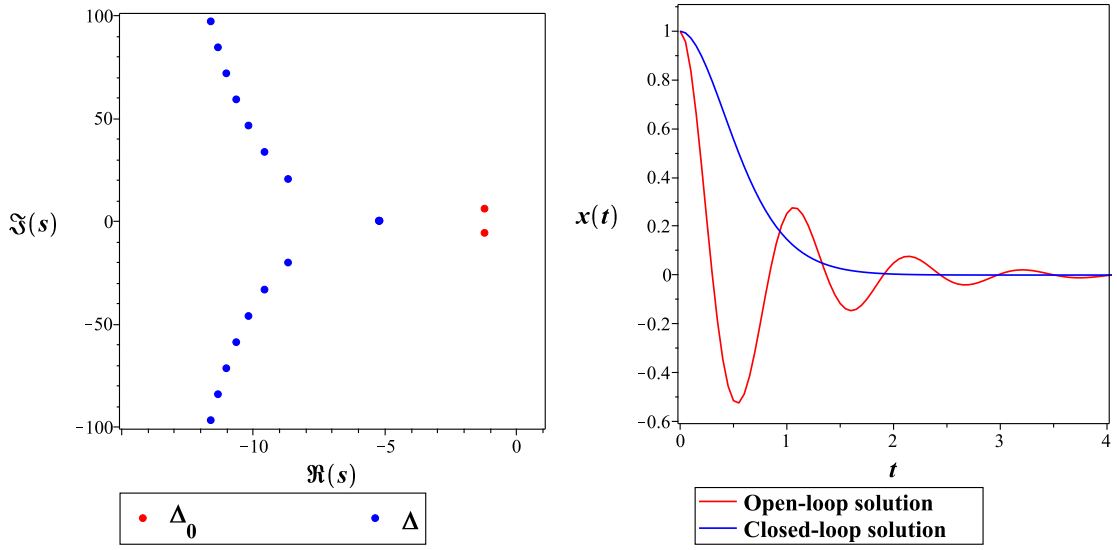

Figure 4.1: (Left): Spectrum distribution of Δ_0 and Δ. (Right): Comparison between the open-loop solution with $u(t) = 0$ and the closed-loop solution for $\xi = 0.2, \omega = 6$ and $\tau = 0.5$.

5 Conclusion

This chapter provided a systematic study of the Multiplicity-Induced-Dominancy (MID) property in parametric second-order retarded differential equations. The latter property enables a delayed-control strategy allowing a proper assignment of the trivial solution's decay rate. Owing to the presented design-algorithm, the method can be easily extended to higher-order systems such as in [29], to neutral systems such as in [7], or even to the design of some classes of partial differential equations such as in [3,10]. Finally, to illustrate its applicability, the damping of oscillations of the standard oscillator has been considered and treated.

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