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► **To cite this version:**

Amira Remadna, Islam Boussaada, Fazia Bedouhene, Silviu-Iulian Niculescu, Azzedine Benchettah. Some Remarks on the Pendulum Stabilization. Multiple Delays & Decay Rate Assignment. 27th International Conference on System Theory, Control and Computing, Oct 2023, Timisoara, Romania. 10.1109/icstcc59206.2023.10308493 . hal-04213496

**HAL Id: hal-04213496**

**<https://centralesupelec.hal.science/hal-04213496v1>**

Submitted on 21 Sep 2023

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# Some Remarks on the Pendulum Stabilization. Multiple Delays & Decay Rate Assignment

Amira Remadna<sup>1</sup>, Islam Boussaada<sup>2,3</sup>, Fazia Bedouhene<sup>4</sup>, Silviu-Iulian Niculescu<sup>2</sup>, Azzedine Benchettah<sup>1</sup>

**Abstract**—A series of recent works have highlighted the interest of multiplicity varieties in the characterization of the exponential decay rate for the solution of linear dynamical systems represented by delayed differential equations. In fact, it has been shown in the case of a single delay that a sufficiently high multiplicity spectral value tends to be dominant, in what is now known as the *multiplicity-induced-dominance* property (MID). Despite the many existing results in such a configuration, only a few concern the case of multiple delays. Through this work, we propose a first characterization of the *MID property* for second-order systems controlled by a two-delay "block". As an application of the results obtained, we consider the problem of stabilization of the classical pendulum with exclusive access to the delayed position.

**Index Terms**—Delay; exponential stability; multiplicity-induced-dominance (MID).

## I. INTRODUCTION

Delays are usually used in modeling a large variety of natural phenomena, from transport and propagation phenomena, signal transmission in communication networks to population dynamics. Further examples can be found in [1]–[5] and the references therein. As mentioned in [6], many methods and tools have been suggested in the literature to address the stability problems (see, e.g., [7]–[10]).

In the particular case of time-invariant (LTI) delay systems (DDE), an interesting and unexpected property, called *multiplicity-induced-dominance* (MID) which, to the knowledge of the authors, has not been sufficiently explored in the literature. Specifically, the MID property simply means that the characteristic root of maximal multiplicity is none other than the *spectral abscissa*<sup>1</sup> of the dynamic system. As indicated in [11], [12], this property opens up a rather interesting perspective from a control point of view by the so-called *partial placement of the poles* method which ensures the location of the remaining roots on the left with respect to the dominant characteristic root in the complex plane. The [13] reference contains many discussions of existing methods for characterizing multiple roots, including also the MID property.

<sup>1</sup>Department of Mathematics, Faculty of sciences, Laboratory of Applied Mathematics, Badji Mokhtar University Annaba, Algeria; amira.remadna@univ-annaba.dz, abenchettah@hotmail.com

<sup>2</sup>Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France; islam.boussaada@l2s.centralesupelec.fr; silviu.niculescu@centralesupelec.fr

<sup>3</sup>IPSA Paris, 94200 Ivry sur Seine, France.

<sup>4</sup>Laboratoire de Mathématiques Pures et appliquées, Mouloud Mammeri University of Tizi-Ouzou, Tizi-Ouzou, Algeria; fazia.bedouhene@ummto.dz

<sup>1</sup>A more in-depth discussion of the spectral abscissa can be found in [3].

The case of the maximal allowed multiplicity<sup>2</sup> of a characteristic root, defined as the *generic MID property*, has been examined in [11], [12] for LTI DDEs including a *single delay* in their models, where a full characterization of such property is given, based on some analytical properties of Kummer and Whittaker confluent hypergeometric functions, which prove to be a powerful tool for the factorization of the corresponding characteristic functions as well as the spectrum location of the non-asymptotic zeros. The work [14] extends and completes the previous ones in the sense that it deals the stabilizing effect of the delay parameter coupled when considering the question of the coexistence of the maximum number of distinct real spectral values.

It should be emphasized that although the method has shown its effectiveness, it has certain shortcomings, in particular, that it cannot be applied systematically to the case of *spectral values with intermediate multiplicity*, which represents a disadvantage of the method. Nevertheless, as discussed in [15], such shortcomings can be overcome using arguments that exploit the structure of the system, the MID property is also valid for spectral values with intermediate multiplicities. Other different arguments have been proposed in [16] for certain classes of unstable systems using tuning PID controller. However, extending the methods in [15], [16] to a more general framework is not a trivial matter. Recently, in [17], a new technique is introduced based on crossing imaginary roots. Such a technique allows to conclude that the MID property holds for scalar delay-differential equations with two delays. Besides the fact that many real-world systems naturally have more than one delay, an important incentive for investigating the stability and stabilization of multi-delay systems is the fact that additional delays can enhance stability conditions, as seen, for example in [17]. Notice, that on beyond of the characterization of the MID property, the contribution [17] provides a method based on the continuous dependency of the spectrum on parameters variation, which can be generalized for investigating the MID property for higher order systems with two-delays.

The purpose of this paper is to address the MID property for a second-order DDE including two delays in the model. More precisely, the contribution of the paper is threefold. First, we provide necessary and sufficient conditions guaranteeing the existence of a spectral value with maximal multiplicity. Second, we recall the steps of the approach described in [17]. The resulting algorithm, which essentially exploits the

<sup>2</sup>i.e., which corresponds to the degree of the quasi-polynomial

analytical properties of the root branches of the normalized quasipolynomial and the behavior of the latter when the two delays are close, will be applied to deduce the MID property for the spectral value with maximal multiplicity. Finally, in order to illustrate the potential of the proposed approach in application, we consider the problem of the stabilization of the classical pendulum with an exclusive access to delayed position.

The remaining paper is organized as follows. Section II includes some important definitions and facts about quasipolynomials and presents the methodology used in the MID property investigation. In Section III, the control problem of stabilization of the classical pendulum with a prescribed decay rate is considered. Section IV is devoted to the statement as well as the proof of the main result. Section V is a direct application of the MID property in the design of delayed proportional controller able in stabilizing the classical pendulum with a prescribed decay rate. The notations are standard.

## II. PREREQUISITES AND PROBLEM STATEMENT

**Definition II.1.** 1) A *quasipolynomial*,  $Q$ , is a complex valued function  $Q : \mathbb{C} \rightarrow \mathbb{C}$  of the form

$$Q(s) = \sum_{j=0}^m P_j(s)e^{\tau_j s}, \quad (1)$$

where  $m$  is a non-negative integer,  $s \in \mathbb{C}$ , and for  $i, j \in \{0, \dots, m\}$ ,  $\tau_i > 0$ ,  $\tau_i \neq \tau_j$  for  $i \neq j$ ,  $P_j : \mathbb{C} \rightarrow \mathbb{C}$  is a nonzero polynomial of degree  $d(P_j) = d_j \geq 0$ .

- 2) The *degree* of the quasi-polynomial  $Q$  is defined as the integer  $D = m + \sum_{j=0}^m d_j$ .
- 3) Let  $\mathcal{S}_\Delta = \{s \in \mathbb{C} \mid \Delta(s) = 0\}$  be the set of spectral values of  $\Delta$ .  $s_0 \in \mathcal{S}_\Delta$  is said to be dominant root of  $\Delta$  if every  $s \in \mathcal{S}_\Delta$ , with  $s \neq s_0$ , satisfies  $\Re(s) < \Re(s_0)$ .

*A. MID property & two delays. An algorithm on a case study*

To the best of the authors' knowledge, the first example exhibiting the MID property in multiple delay case has been reported in [18] where the stabilization of the inverted pendulum has been considered using two delays proportional signals. A recent result [17], considers the scalar DDE:

$$y'(t) + a_0 y(t) - a_1 y(t - \tau_1) - a_2 y(t - \tau_2) = 0, \quad (2)$$

whose characteristic quasipolynomial is the function  $\Delta : \mathbb{C} \rightarrow \mathbb{C}$  defined, for  $s \in \mathbb{C}$ , by

$$\Delta(s) = s + a_0 - a_1 e^{-s\tau_1} - a_2 e^{-s\tau_2}. \quad (3)$$

with a degree equal to  $D = 3$ . Notice, that on beyond of the characterization of the MID property for equation (2), the contribution [17] provides a method based on the continuous dependency of the spectrum on parameters variation, which can be generalized for investigating the MID property for higher order systems with two-delays.

From [17, Corollary 2.2] this implies that a root  $s_0$  of  $\Delta$  with multiplicity 3 has maximal multiplicity. In this section,

we aim to describe the MID methodology that provides a choice of  $s_0$ ,  $a_1$ , and  $a_2$  ensuring that the quasipolynomial  $\Delta(\cdot)$  satisfies the MID property, in which case we also have the exact expression of the spectral abscissa. Without loss of generality, we can assume that  $0 < \tau_1 < \tau_2$ . Let  $\lambda = \frac{\tau_1}{\tau_2} \in (0, 1)$ .

Here we recall the steps of the approach described in [17] which is based on the continuous dependency of the spectrum with respect to parameters changes and can be summarized as follows. Once the parameters' values guaranteeing the existence of a spectral value with the maximal multiplicity are established, one studies the limiting parameters values. More precisely, one needs first to show the MID property to holds for  $\lambda = 0$  as well as  $\lambda = 1$ . Next, it suffices to prove the uniform boundedness of the spectral value branch in all compact sets of the interval  $(0, 1)$ . Finally, one carries out the spectral value branch tendency. If this latter is of constant sign, thanks to the continuous dependency of the spectrum on parameters' variation, this allows to conclude the MID property to hold for any  $\lambda \in [0, 1]$ .

Before describing the steps of the algorithm in question, let us first recall the definition of a branch of roots.

**Definition II.2** ([17]). Let  $Q(\cdot, \lambda)$ ,  $\lambda \in (0, 1)$ , be a family of quasipolynomials. A *branch of roots of  $Q$*  is a function  $s : I \rightarrow \mathbb{C}$  defined on an open interval  $I \subset (0, 1)$ , such that  $s(\lambda)$  is a root of  $Q(\cdot, \lambda)$  for every  $\lambda \in I$ .

### Algorithm Description

#### Step 1: Forcing multiplicity

Choice of the real numbers  $s_0$ ,  $a_1$  and  $a_2$  ensuring the existence of a spectral value  $s_0$  with maximal multiplicity. This latter is given by:

$$s_0 = -a_0 - \frac{1}{\tau_1} - \frac{1}{\tau_2}, \quad a_1 = \frac{-\tau_2 e^{s_0 \tau_1}}{\tau_1(\tau_2 - \tau_1)}, \quad a_2 = \frac{\tau_1 e^{s_0 \tau_2}}{\tau_2(\tau_2 - \tau_1)}. \quad (4)$$

#### Step 2: Normalization of the quasipolynomial $Q$

Define the normalized quasipolynomial  $Q(\cdot, \lambda) : \mathbb{C} \rightarrow \mathbb{C}$  by

$$Q(s, \lambda) = \tau_2 \Delta(s_0 + \frac{s}{\tau_2})$$

for  $s \in \mathbb{C}$ , and express this quantity as a function of  $s$  and  $\lambda = \tau_1/\tau_2$ , as follows:

$$Q(s, \lambda) = s - \frac{\lambda + 1}{\lambda} - \frac{\lambda}{1 - \lambda} e^{-s} + \frac{1}{\lambda(1 - \lambda)} e^{-\lambda s}. \quad (5)$$

#### Step 3: The MID property holds in the limiting $\lambda$ values

This step consists in evaluating the quasipolynomial in the limiting values of the parameter  $\lambda$  and to check the MID to holds at these values. As a matter of fact,

$$\lim_{\lambda \rightarrow 1} Q(s, \lambda) = (s + 2) e^{-s} + s - 2$$

where it has been shown in [19] that the MID is verified thanks to the integral representation of the resulting quasipolynomial

in terms of Kummer hypergeometric function, see also [12]. Next,

$$\lim_{\lambda \rightarrow 0} Q(s, \lambda)/\lambda = \frac{s^2}{2} - s + 1 - e^{-s}$$

where the MID has been shown in [20].

**Step 4: Existence of an analytic branch**  $s : I \subset (0, 1) \rightarrow \mathbb{C}$

Let  $s_0$  be of root of  $Q(\cdot, \lambda)$ , with  $\Re(s_0) \geq 0$  and  $s_0 \neq 0$ . Let  $s : I \rightarrow \mathbb{C}$  be an analytic branch of roots of  $Q$ , such that  $s_0 \in I$  and  $s(\lambda_0) = s_0$ . Compute

$$s'(\lambda_0) = -\frac{\partial_\lambda \tilde{Q}(s(\lambda_0), \lambda_0)}{\partial_s \tilde{Q}(s(\lambda_0), \lambda_0)} \quad (6)$$

where

$$\tilde{Q}(s, \lambda) = \lambda(1-\lambda)Q(s, \lambda) = \lambda(1-\lambda)s - 1 + \lambda^2 - \lambda^2 e^{-s} + e^{-\lambda s}. \quad (7)$$

for every  $s \in \mathbb{C}$  and  $\lambda \in (0, 1)$ . The numerator and denominator of (6) are given by:

$$\partial_s \tilde{Q}(s, \lambda) = \lambda(1-\lambda) + \lambda^2 e^{-s} - \lambda e^{-\lambda s}, \quad (8)$$

$$\partial_\lambda \tilde{Q}(s, \lambda) = (1-2\lambda)s + 2\lambda - 2\lambda e^{-s} - s e^{-\lambda s}. \quad (9)$$

**Step 5: Boundedness of the analytic function**  $s$

Let  $K$  be a compact subset of  $(0, 1)$ . Let  $\alpha = \inf\{\Re(s(\lambda)) \mid \lambda \in I\}$ ,  $\lambda_{\min} = \inf K$ , and  $\lambda_{\max} = \sup K$ . Use  $Q(s(\lambda), \lambda) = 0$  together with (5) to obtain, for every  $\lambda \in I \cap K$ .

$$|s(\lambda)| \leq \frac{\lambda_{\min} + 1}{\lambda_{\min}} + \frac{\lambda_{\max} e^{-\alpha}}{1 - \lambda_{\max}} + \frac{e^{\lambda_{\max}|\alpha|}}{\lambda_{\min}(1 - \lambda_{\max})}.$$

The latter quantity only depends on  $K$  and  $\alpha$ .

**Step 6: Constance of the spectral value tendency**

Let  $s : I \rightarrow \mathbb{C}$  be an analytic branch of roots of  $Q$  and  $\lambda_0 \in I$  be such that  $\Re(s(\lambda_0)) = 0$  and  $s(\lambda_0) \neq 0$ . Then

$$\Re(s'(\lambda_0)) > 0. \quad (10)$$

In the contrary case, let  $\omega = \Im(s(\lambda_0))$ , and note that  $s(\lambda_0)$  is a simple root of  $Q(\cdot, \lambda_0)$ , thus by taking the real and imaginary parts, we have that  $Q(i\omega, \lambda_0) = 0$  if and only if

$$\cos(\lambda_0 \omega) - \lambda_0^2 \cos(\omega) + \lambda_0^2 - 1 = 0, \quad (11a)$$

$$\sin(\lambda_0 \omega) - \lambda_0^2 \sin(\omega) + \lambda_0^2 \omega - \lambda_0 \omega = 0. \quad (11b)$$

1) From (6) and (11) one gets:

$$\text{sign}(\Re(s'(\lambda_0))) = \text{sign}(H(\omega, \lambda_0)), \quad (12)$$

where

$$\begin{aligned} H(\omega, \lambda_0) &= -R_1(\omega, \lambda_0)R_2(\omega, \lambda_0) - I_1(\omega, \lambda_0)I_2(\omega, \lambda_0) \\ &= 2\lambda_0^3(1 - \lambda_0)(\omega \cos(\omega/2) - 2 \sin(\omega/2))^2. \end{aligned}$$

and

$$\begin{cases} I_1(\omega, \lambda_0) = \Im(\partial_s \tilde{Q}(i\omega, \lambda_0)) \\ I_2(\omega, \lambda_0) = \Im(\partial_\lambda \tilde{Q}(i\omega, \lambda_0)) \\ R_1(\omega, \lambda_0) = \Re(\partial_s \tilde{Q}(i\omega, \lambda_0)) \\ R_2(\omega, \lambda_0) = \Re(\partial_\lambda \tilde{Q}(i\omega, \lambda_0)) \end{cases} \quad (13)$$

and thus  $H(\omega, \lambda_0) = 0$  iff

$$2(\lambda_0 + 1)(1 - \cos(\omega)) - (1 - \lambda_0)\omega^2 - 2 \sin(\omega)\omega\lambda_0 = 0 \quad (14)$$

2) Solve (14) with respect to  $\lambda$ , and show that  $\lambda = 1$ .

### III. CONTROLLING OSCILLATIONS USING DELAYS

#### A. Control and optimization problem

Consider the delay-differential equation

$$y''(t) + a_1 y'(t) + a_0 = u(t) \quad (15)$$

where  $u$  is the unknown control. Assume that the system (15) is unstable in the uncontrolled case, namely when  $u(t) = 0$ . Our goal is to build an appropriate delayed state feedback controller of the form

$$u(t) = -b_1 y(t - \tau_1) - b_2 y(t - \tau_2) \quad (16)$$

guaranteeing the stability of system (15) in closed-loop:

$$y''(t) + a_1 y'(t) + a_0 + b_1 y(t - \tau_1) + b_2 y(t - \tau_2) = 0. \quad (17)$$

The characteristic quasi-polynomial function corresponding to the closed-loop system is described as follows:

$$\Delta(s) = s^2 + a_1 s + a_0 + b_1 e^{-s\tau_1} + b_2 e^{-s\tau_2} = 0 \quad (18)$$

From [17, Corollary 2.2], we deduce that a root  $s_0$  of  $\Delta$  with multiplicity 4 has maximal multiplicity. The following theorem, which constitutes the main result of this paper, characterizes the values of  $s_0$ ,  $a_1$ ,  $b_1$  and  $b_2$  guaranteeing that the quasipolynomial  $\Delta(\cdot)$  satisfies the multiplicity-induced-dominancy property, in which case the exact expression of the spectral abscissa is also obtained.

Using the following change of variables  $\lambda = \frac{\tau_1}{\tau_2} \in (0, 1)$  one obtains the corresponding normalized characteristic function

$$\begin{aligned} Q_2(s, \lambda) &:= \tau_2^2 \Delta(s_0 + \frac{s}{\tau_2}) \\ &= s^2 + \alpha_1 s + \alpha_0 + \beta_1 e^{-\lambda s} + \beta_2 e^{-s} \end{aligned} \quad (19)$$

where

$$\alpha_1 = \tau_2(2s_0 + a_1), \quad \alpha_0 = \tau_2^2(s_0^2 + a_1 s_0 + a_0) \quad (20a)$$

$$\beta_1 = \tau_2^2 b_1 e^{-s_0 \tau_1}, \quad \beta_2 = \tau_2^2 b_2 e^{-s_0 \tau_2}. \quad (20b)$$

Note that the mapping

$$S(\Delta) \rightarrow S(Q_2); \quad s \mapsto \tau_2^2(s - s_0) \quad (21)$$

is a bijection from roots of  $\Delta$  to roots of  $Q_2(\cdot, \lambda)$ , for every  $\lambda \in (0, 1)$ , preserving their multiplicities and the MID property (see [17]).

#### IV. STATEMENT OF THE MAIN RESULT

In this section, we present the main result of the paper, which consists in designing a feedback law (16) with two delays, for system (15).

**Theorem IV.1.** *Let  $a_0, a_1, b_1, b_2$  and  $s_0$  be real numbers and  $\tau_1, \tau_2$  be positive real numbers with  $\tau_1 \neq \tau_2$ . The closed-loop system (17) admits  $s_0$  as a spectral value with maximal multiplicity 4 if and only if*

$$\begin{cases} s_0 = -\frac{1}{2}a_1 - \frac{1}{\tau_1} - \frac{1}{\tau_2} \\ a_0 = \frac{s_0^2\tau_1^2\tau_2^2 + 2s_0\tau_1\tau_2^2 + 2s_0\tau_1^2\tau_2 + 2\tau_2^2 + 2\tau_1^2 + 2\tau_1\tau_2}{\tau_2^2\tau_1^2} \\ b_1 = -\frac{2e^{s_0\tau_1}\tau_2}{\tau_1^2(\tau_2 - \tau_1)} \\ b_2 = \frac{2e^{s_0\tau_2}\tau_1}{\tau_2^2(\tau_2 - \tau_1)}. \end{cases} \quad (22)$$

Furthermore, in that case,  $s_0$  is the spectral abscissa of system (17) and the MID property holds.

##### A. Proof of Theorem IV.1

The proof of the theorem will follow the steps of the methodology outlined above.

##### Step 1: Forcing multiplicity

The calculation of the coefficients given by (22) is obtained by solving an algebraic system of four equations with four variables  $s_0, a_0, b_1$  and  $b_0$  when  $\tau_1, \tau_2, a_1$  are fixed.

##### Step 2: Normalization of the quasi-polynomial $Q_2$

Under (20) and (22), the quasi-polynomial  $Q_2$  from (19) reads:

$$Q_2(s, \lambda) = s^2 - \frac{2(\lambda + 1)}{\lambda}s + \frac{2(\lambda^2 + \lambda + 1)}{\lambda^2} - \frac{2}{\lambda^2(1 - \lambda)}e^{-\lambda s} + \frac{2\lambda}{1 - \lambda}e^{-s} \quad (23)$$

##### Step 3: The MID property holds in the limiting $\lambda$ values

This step consists in evaluating the quasipolynomial in the limiting values of the parameter  $\lambda$  and to check the MID to holds at these values. In our case,

$$Q^1(s) := \lim_{\lambda \rightarrow 1} Q(s, \lambda) = (-2s - 6)e^{-s} + s^2 - 4s + 6$$

where it has been shown in [12] that the MID is verified since the obtained quasipolynomial,  $Q^1$  shares its zeros with an appropriate Kummer hypergeometric function, see also [21]. Next,

$$Q^0(s) := \lim_{\lambda \rightarrow 0} Q(s, \lambda)/\lambda = \frac{s^3}{3} - s^2 + 2s - 2 + 2e^{-s}$$

where the MID has been shown in [12] for  $Q^0$ .

##### Step 4: Existence of an analytic branch $s : I \subset (0, 1) \rightarrow \mathbb{C}$

Consider  $\tilde{Q}_2 : \mathbb{C} \times (0, 1) \rightarrow \mathbb{C}$  defined by  $\tilde{Q}_2(s, \lambda) = \lambda^2(1 - \lambda)Q(s, \lambda)$ , namely

$$\tilde{Q}_2(s, \lambda) = \lambda^2(1 - \lambda)s^2 - 2\lambda(1 - \lambda^2)s + 2 - 2\lambda^3 + 2\lambda^3e^{-s} - 2e^{-\lambda s} \quad (24)$$

The existence of the above mentioned branch of roots  $s$ , satisfying  $s(\lambda_0) = s_0$ , and

$$s'(\lambda_0) = -\frac{\partial_\lambda \tilde{Q}_2(s(\lambda_0), \lambda_0)}{\partial_s \tilde{Q}_2(s(\lambda_0), \lambda_0)} \quad (25)$$

is an immediate consequence of the simplicity of the nontrivial root  $s$  and the implicit function theorem for analytic functions (see, e.g., [22, Theorem 2.1.2]).

##### Step 5: Boundedness of the analytic function $s$

Let  $\alpha = \inf\{\Re(s(\lambda)) \mid \lambda \in I\}$ ,  $\lambda_{min} = \inf K$ , and  $\lambda_{max} = \sup K$ . Then, using the fact that  $Q_2(s(\lambda), \lambda) = 0$  we get, for every  $\lambda \in I \cap K$ ,

$$s(\lambda) \leq \left( \frac{2(\lambda_{min} + 1)}{\lambda_{min}} |\alpha| + \frac{2(\lambda_{min}^2 + \lambda_{min} + 1)}{\lambda_{min}} + \frac{2e^{\lambda_{max}|\alpha|}}{\lambda_{min}^2(1 - \lambda_{max})} + \frac{2\lambda_{max}}{1 - \lambda_{max}}e^{-\alpha} \right)^{\frac{1}{2}}$$

The latter quantity only depends on  $K$  and  $\alpha$ , and thus provides a bound  $s$ .

##### Step 6: Constance of the spectral value tendency

Let us show that any analytic branch of roots of  $Q_2$  and any  $\lambda_0 \in I$  satisfying  $\Re(s(\lambda_0)) = 0$  and  $s(\lambda_0) \neq 0$ , have the property

$$\Re(s'(\lambda_0)) > 0. \quad (26)$$

Let  $\omega = \Im(s(\lambda_0))$ . Note that  $s(\lambda_0)$  is a simple root of  $Q_2(\cdot, \lambda_0)$ . We have in particular  $Q_2(i\omega, \lambda_0) = 0$ . This later is equivalent to

$$\lambda_0^2\omega^2 - \lambda_0^3\omega^2 - 2 + 2\lambda_0^3 + 2\cos(\lambda_0\omega) - 2\lambda_0^3\cos(\omega) = 0 \quad (27a)$$

$$2\lambda_0^3\sin(\omega) - 2\sin(\lambda_0\omega) + 2\lambda_0\omega - 2\lambda_0^3\omega = 0 \quad (27b)$$

Let  $\tilde{Q}_2$  be defined as in (24). Let's exploit the numerator and the denominator in (25).

- **Computation of  $\partial_s \tilde{Q}_2(i\omega, \lambda_0)$ .** For  $s = i\omega, \lambda = \lambda_0$ , one obtains from (24) that

$$\begin{aligned} \partial_s \tilde{Q}_2(i\omega, \lambda_0) &= 2\lambda_0^2(1 - \lambda_0)i\omega - 2\lambda_0(1 - \lambda_0^2) \\ &\quad + 2\lambda_0(\cos(\lambda_0\omega) - i\sin(\lambda_0\omega)) \\ &\quad - 2\lambda_0^3(\cos(\omega) - i\sin(\omega)) \end{aligned} \quad (28)$$

Putting  $R_1(\omega, \lambda_0) := \Re(\partial_s \tilde{Q}_2(i\omega, \lambda_0))$  and using (27a), we compute

$$R_1(\omega, \lambda_0) = -\lambda_0^3(1 - \lambda_0)(2\cos(\omega) + \omega^2 - 2) \quad (29)$$

In the same way, setting  $I_1(\omega, \lambda_0) = \Im(\partial_s \tilde{Q}_2(i\omega, \lambda_0))$  and using (27b), we have

$$I_1(\omega, \lambda_0) = -2\lambda_0^3(1 - \lambda_0)(\omega - \sin(\omega)) \quad (30)$$

- **Computation of  $\partial_\lambda \tilde{Q}_2(i\omega, \lambda_0)$ .** For  $s = i\omega$ ,  $\lambda = \lambda_0$ ,

$$\begin{aligned} \partial_\lambda \tilde{Q}_2(i\omega, \lambda_0) &= (3\lambda_0^2 - 2\lambda_0)\omega^2 + (6\lambda_0^2 - 2)i\omega - 6\lambda_0^2 \\ &+ 2i\omega(\cos(\lambda_0\omega) - i\sin(\lambda_0\omega)) + 6\lambda_0^2(\cos(\omega) - i\sin(\omega)) \end{aligned} \quad (31)$$

Setting  $R_2(\omega, \lambda_0) = \Re(\partial_\lambda \tilde{Q}_2(i\omega, \lambda_0))$  and using (27b), then

$$\begin{aligned} R_2(\omega, \lambda_0) &= -6\lambda_0^2 + 6\lambda_0^2 \cos(\omega) + 2\omega\lambda_0^3 \sin(\omega) \\ &+ 3\lambda_0^2\omega^2 - 2\lambda_0^3\omega^2 \end{aligned} \quad (32)$$

Similarly, setting  $I_2(\omega, \lambda_0) = \Im(\partial_\lambda \tilde{Q}_2(i\omega, \lambda_0))$  and using (27b), we have

$$\begin{aligned} I_2(\omega, \lambda_0) &= 6\lambda_0^2(\omega - \sin(\omega)) + 2\lambda_0^3\omega(\cos(\omega) - 1) \\ &- \lambda_0^2\omega^3(1 - \lambda_0) \end{aligned} \quad (33)$$

- **Evaluation of the sign of the real part of  $s'(\lambda_0)$**  Since

$$\Re(s'(\lambda_0)) = \frac{R_1(\omega, \lambda_0)R_2(\omega, \lambda_0) + I_1(\omega, \lambda_0)I_2(\omega, \lambda_0)}{-(R_1^2(\omega, \lambda_0) + I_1^2(\omega, \lambda_0))} \quad (34)$$

and taking into account (29), (30), (32), and (33), we have

$$\begin{aligned} &- R_1(\omega, \lambda_0)R_2(\omega, \lambda_0) - I_1(\omega, \lambda_0)I_2(\omega, \lambda_0) \\ &= \lambda_0^5(1 - \lambda_0)(-24\omega \sin(\omega) + 12\omega^2 \cos(\omega) + 24 \\ &- 24 \cos(\omega) + \omega^4 + 2\omega^3 \sin(\omega)) \end{aligned} \quad (35)$$

The variations of the function  $\omega \mapsto (-24\omega \sin(\omega) + 12\omega^2 \cos(\omega) + 24 - 24 \cos(\omega) + \omega^4 + 2\omega^3 \sin(\omega))$  give that  $\Re(s'(\lambda_0)) \geq 0$ . In order to prove that this inequality is strict, simple computations lead to:

$$\begin{aligned} &4\omega^2 - 8 - 4\lambda_0^2\omega^2 \cos(\omega)(1 - \lambda_0) + 8 \cos(\omega) + \lambda_0\omega^4 \\ &- 2\lambda_0^2\omega^4 + 4\lambda_0^2\omega^2 + \lambda_0^3\omega^4 - 8\lambda_0^3 \cos(\omega) + 8\lambda_0^3 \\ &- 8\lambda_0\omega^2 + 8\lambda_0\omega \sin(\omega) - 8\lambda_0^3\omega \sin(\omega) = 0. \end{aligned} \quad (36)$$

Solving (36) with respect to  $\lambda_0$ , we deduce that  $\lambda_0 = 1$  is the unique real solution of (36), which gives the contradiction. Thus, (26) is proved.

The proof of Theorem IV.1 is achieved.

## V. STABILIZING THE PENDULUM VIA DELAY ACTION

As an illustration of the main result of the present work, we revisit the stabilization of the friction-free dynamical modeling of the classical pendulum. The adopted model is studied in [12] in the context of single delay MID-based stabilization and, in the sequel, we keep the same notations. The controlled classical pendulum is governed by the following second-order differential equation

$$\ddot{\theta}(t) + \frac{g}{L} \sin(\theta(t)) = u(t) \quad (37)$$

where  $\theta(t)$  stands for the angular displacement of the pendulum at time  $t$  with respect to the stable equilibrium position,  $L$  is the length of the pendulum,  $g$  is the gravitational acceleration, and  $u(t)$  represents the control law, which stems from an

applied external torque. Our aim is to improve the exponential decay rate of the open-loop stable equilibrium via the action of the external torque  $u$ . This suggests first to linearize (37) around the trivial equilibrium and get

$$\ddot{\theta}(t) + a_0 \theta(t) = u(t), \quad (38)$$

where  $a_0 = \frac{g}{L}$ . Next assume that such a system is controlled using a delayed Proportional controller of the form

$$u(t) = -b_1 \theta(t - \tau_1) - b_2 \theta(t - \tau_2), \quad (39)$$

with  $(b_1, b_2) \in \mathbb{R}^2$ , which has the advantage of avoiding noises due to differentiations. The local stability of the closed-loop system is reduced to study the location of the spectrum of the quasipolynomial

$$\Delta(s) = s^2 + a_0 + b_1 e^{-\tau_1 s} + b_2 e^{-\tau_2 s}, \quad (40)$$

which is a quasipolynomial of degree 4. Here, the horizontal control force will be exclusively referred to as position feedback, exploiting multiple delayed measurements instead of derivative terms, as has been already suggested in [18] where the problem of stabilization of the inverted pendulum has been considered but using other control strategy relying on spectral projections on the center-manifold.

Since the delays  $\tau_1$  and  $\tau_2$  are control parameters, thanks to Theorem IV, and in particular the expressions of  $s_0$  and  $a_0$ , it follows that the delays have to be chosen according to the strategy below: first one has to select a delay  $\tau_1$  satisfying

$$\tau_1 > \sqrt{\frac{L}{g}}, \quad (41)$$

then one substitutes the chosen value in the expression of  $\tau_2$  given by

$$\tau_2 = \frac{\tau_1}{\sqrt{\frac{g\tau_1^2 - L}{L}}} \quad (42)$$

lastly, the controllers gains are then to be tuned such that:

$$b_1 = \frac{2\tau_2}{\tau_1^2 (\tau_1 - \tau_2) e^{\frac{\tau_1 + \tau_2}{\tau_2}}}, b_2 = -\frac{2\tau_1}{\tau_2^2 e^{\frac{\tau_1 + \tau_2}{\tau_1}} (\tau_1 - \tau_2)} \quad (43)$$

Notice that this tuning allows to the closed-loop characteristic function to admit a spectral abscissa as the quadruple root

$$s_0 = -\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}, \quad (44)$$

then, as illustrated in Figure 1, the exponential decay rate of the closed-loop system is given by  $\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}$ .

## VI. CONCLUSION AND PERSPECTIVES

The MID property for a second-order scalar equation with two delays has been examined in this paper. Necessary and sufficient conditions are provided for a spectral value to have maximum multiplicity. In order to deduce the MID property for the second-order system, we applied the perturbative approach described in [17], which exploits the analytical properties of the branches of roots of the normalized quasipolynomial and the behavior of the latter when the two

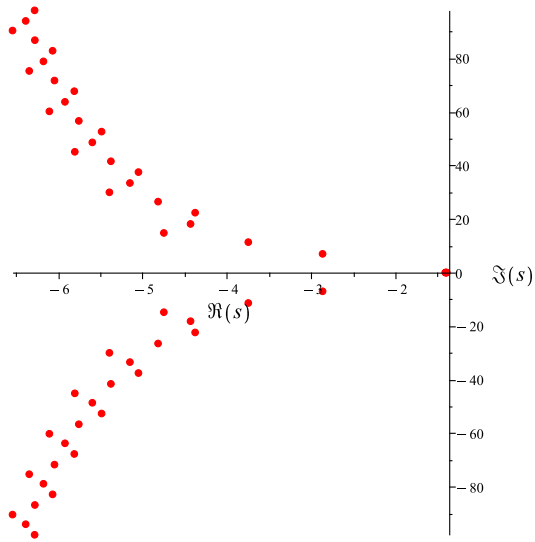


Figure 1. Spectrum distribution of the closed-loop system (38)-(39) where the delays are chosen such that  $\tau_1$  satisfies the constraint (41) and  $\tau_2$  satisfies (42) and controller's gains satisfy (43). The numerical values are taken as follows: the ratio  $L/g = 1$ , the delays  $(\tau_1, \tau_2) = (5/4, 5/3)$ . The spectral abscissa is located at the quadruple root  $s_0 = -7/5$  derived from (44).

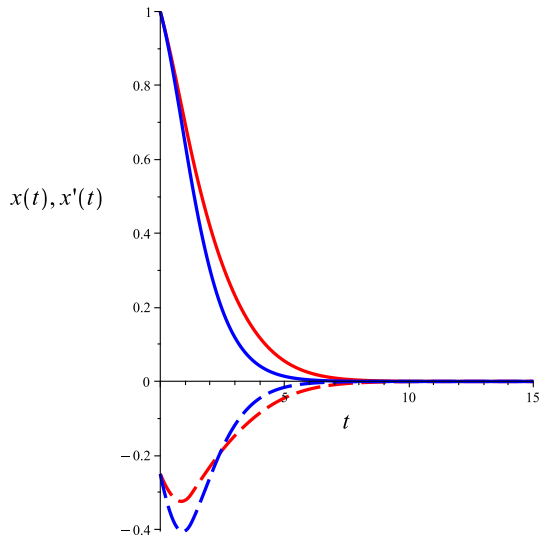


Figure 2. In blue (red), the time response of the closed-loop linear (non-linear) system solution  $x(t)$  (solid) and  $x'(t)$  (dashed) (38)-(39), respectively (37)-(39). The initial conditions are  $(x(0), x'(0)) = (1, -1/4)$  and the numerical values are the same as in Figure 1.

delays are close. Finally, we illustrated the potential of the proposed approach by considering the problem of stabilizing the classical pendulum with an exclusive access to the delayed position.

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