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Model order reduction for linear time delay systems: a delay-dependent approach based on energy functionals ⋆

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Abstract

This paper proposes a model order reduction technique for asymptotically stable linear time delay systems with pointwise delays. The presented delay-dependent approach, which can be regarded as an extension of existing balancing model order reduction techniques for linear delay-free systems, is based on energy functionals that characterize observability and controllability properties of the time delay system. The reduced model obtained by this approach is an asymptotically stable time delay system of the same type as the original model, meaning that the approach is both stability- and structure-preserving. It also provides an a priori bound on the reduction error, serving as a measure of the reduction accuracy. The effectiveness of the proposed method is illustrated by numerical simulations.

1 Introduction

Engineering systems such as drilling systems, traffic systems and electric circuits, as well as phenomena in economics and biology, can often be described by models in terms of delay differential equations (Erneux, 2009; Kolmanovskii and Myshkis, 1992). For complex engineering systems, however, such models might be of high order, i.e., described in terms of a high number of state variables, which complicates analysis and may prohibit the design of controllers. For instance, robust control techniques in, e.g., (Gumussoy and Michiels, 2011) can be applied effectively only to low-order delay systems. To address these issues of model complexity, this paper presents a method for model reduction of linear time delay systems.

For the problem of model order reduction of systems in terms of ordinary differential equations, many techniques, such as balanced truncation (Moore, 1981), have been proposed over the past four decades (for an overview, see (Antoulas, 2005)). Model order reduction techniques for systems in terms of delay-differential equations have also been considered, mainly by extending those of delay-free systems. We may split these into two main categories: 1) methods approximating the time delay system by a low-order finite-dimensional model, and 2) structure-preserving methods that preserve the infinite-dimensional nature of the delay system. The majority of the existing methods are of the first category, as analysis and design based on a finite-dimensional model enables the use of classical well-developed techniques (e.g., for controller design). The Padé approximation has probably been the most popular method for finite-dimensional approximation of delay systems (Lam, 1993). After a Padé approximation, the resulting finite-dimensional model can be further reduced using conventional model order reduction methods. In (Michiels et al., 2011), such a finite-dimensional approximation is obtained by performing a spectral discretization and using Krylov subspace projection. Methods based on series expansions (Glover et al., 1988; Makila and Partington, 1999), including the Padé approximation as a special case, and formulating the model reduction problem as a $\mathcal{H}_\infty$- or $\mathcal{H}_2$-norm optimization problem (Xu et al., 2001; Duff et al., 2015) are other examples from the first category.

In this paper, we are interested in infinite-dimensional, but low-order, model approximations, because for a given order of the reduced model, a reduced model in terms of delay differential equations has in general the potential to be more accurate than a finite-dimensional approximation of the same order (van de Wouw et al.,

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2015). Moreover, the preservation of the delay nature enables the preservation of some other desirable properties of the model such as stability (Richard, 2003; Gu et al., 2003). As another example, wave propagation effects (Aarsnes and van de Wouw, 2018) can often be captured through delays that should be preserved in a low-complexity model. In addition, powerful analysis and controller design techniques are available today for time delay systems, e.g., (Michiels and Niculescu, 2014).

We can further divide structure-(i.e. delay-)preserving reduction methods into two major groups. Firstly, methods exist that preserve the model structure, but not necessarily stability properties of the high-order model. As a result, these methods usually lack a measure on their accuracy, e.g., an error bound. Among these are position balancing (Jarlebring et al., 2013) and methods that are developed based on series expansions and Krylov subspace projection, such as Laguerre expansion (Wang et al., 2016) and Laurent series expansion, where the latter is closely related to moment matching for time delay systems (Scarciotti and Astolfi, 2016). The second group of methods, in which the contribution of the current paper also belongs, not only preserves the model structure, but also preserves stability properties and provides computable and guaranteed error bounds. The method proposed in (Xu et al., 2001) is one such method. However, it is a delay-independent approach applicable to a limited class of systems. Moreover, it can lead to conservative model approximations for decreasing delays. A delay-dependent variant of this method can be found in (Lam et al., 2005). In practice, the applicability of the methods in (Xu et al., 2001; Lam et al., 2005) is however limited to delay systems of low order, as the reduction procedure relies on the solution of sets of non-convex matrix inequalities. The method proposed in (van de Wouw et al., 2015) provides an alternative perspective by decomposing the delay system into a feedback interconnection of a high-order delay-free subsystem and a low-dimensional delay-dependent operator, and employing a conventional model order reduction method to reduce the system by reducing only the delay-free subsystem. This method generally leads to conservative results (especially for increasing delays), as it relies on the satisfaction of a small-gain condition. Moreover, it is effective if the delay effects are local, in the sense that the delay only affects a lower-dimensional part of the state-variables in the right-hand side of the delay differential equation. The proposed method in the current paper is not limited by such restrictions.

In this paper, inspired by balanced truncation for finite-dimensional systems, we define energy functionals that provide a measure of observability and controllability of delay systems. However, the exact computation and characterization of these functionals is challenging, motivating the definition of computable delay-dependent functionals which, as a contribution of this paper, are shown to bound the energy functionals. The delay dependency of these bounds, which are in the form of Lyapunov-Krasovskii functionals (Fridman, 2014), makes these tighter than delay-independent variants. Characterized by the solution to matrix inequalities, these quadratic bounds are used to perform a balancing transformation to sort the state components of the delay system according to their relative importance from an input-output perspective.

The main contribution of this work is the development of a delay-dependent model reduction method for time delay systems, endowed by taking into account the size of the delay during the balancing procedure, as an extension to preliminary delay-independent results in (Besselink et al., 2017). The main benefits of this delay-dependent extension are to, first, enlarge the class of time delay systems that can be reduced and, second, to reduce the typically large conservatism of the delay-independent results for small delays, without sacrificing the performance for large delays. We will prove that the presented model order reduction method preserves both asymptotic stability and the infinite-dimensional nature of the time delay system while also providing an a priori computable, delay-dependent error bound. This error bound represents a measure of the accuracy of the model approximation, and it can be used in, e.g., robustness analyses and design of robust controllers.

Outline. After introducing notation, a problem statement is given in Section 2, whereas Section 3 introduces and characterizes the observability and controllability energy functionals. Section 4 is devoted to the description of the proposed delay-dependent model order reduction procedure. A numerical example is presented in Section 5 and conclusions are presented in Section 6.

Notation. Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ refer to the fields of real and complex numbers, respectively. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x| = \sqrt{x^T x}$. The space of all functions $x : [a, b] \to \mathbb{R}^n$ with bounded norm $\|x\|_2^2 = \int_a^b |x(t)|^2 \, dt$ is denoted by $\mathcal{L}_2([a, b], \mathbb{R}^n)$, whereas $\mathcal{L}_\infty([a, b], \mathbb{R}^n)$ indicates the space of all bounded piecewise continuous functions mapping $[a, b]$ into $\mathbb{R}^n$. The notation $\mathcal{C}_n = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ refers to the Banach space of absolutely continuous functions that map the interval $[-\tau, 0]$ into $\mathbb{R}^n$. Moreover, $\mathcal{W}_n = \mathcal{W}([-\tau, 0], \mathbb{R}^n)$ refers to the space of functions $\varphi \in \mathcal{C}_n$ with square-integrable derivative, i.e., $\dot{\varphi} \in \mathcal{L}_2([-\tau, 0], \mathbb{R}^n)$ for $\varphi \in \mathcal{W}_n$ (Kolmanovskii and Myshkis, 1992). A block-diagonal matrix with $A_1, ..., A_m$ on the diagonal is represented as $\text{blkdiag}\{A_1, \cdots, A_m\}$, and I_m denotes the $m \times m$ identity matrix. The transpose and conjugate transpose of a matrix A are denoted by A^T and A^H, respectively. Finally, a star $*$ in a symmetric matrix represents a symmetric term.

2 Problem statement

Consider a time delay system Ω with point-wise delay in the state variables as

$$\Omega : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t), \\ y(t) = Cx(t) + C_d x(t-\tau) + Du(t), \\ x_0 = \varphi, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state vector, input and output, respectively, and τ denotes a constant time delay. We assume there exists a constant $\bar{\tau} > 0$ such that for each $\tau \in [0, \bar{\tau}]$, the system is asymptotically stable for zero input. For $t \in \mathbb{R}$, the function

segment $x_t : [-\tau, 0] \to \mathbb{R}^n$ denotes the state of Ω at the time instance t with $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$. The initial condition is given by φ, such that $x(t) = \varphi(t)$, $t \in [-\tau, 0]$.

The objective is to find a reduced-order model $\hat{\Omega}$ that closely approximates the input-output behaviour of Ω. We emphasize that due to the fact that the state belongs to $\mathcal{C}_n$, the system Ω has an infinite-dimensional nature in addition to the finite number of dynamical equations describing it. In this paper, model order reduction is accomplished with regard only to the latter aspect, i.e., by reducing the number of the dynamical equations of Ω. In particular, $\hat{\Omega}$ should have the following characteristics:

- $k < n$, with k the order of the reduced-order model $\hat{\Omega}$;
- the infinite-dimensional nature of Ω is preserved in $\hat{\Omega}$, i.e. $\hat{\Omega}$ is also a time delay system;
- for each $\tau \in [0, \bar{\tau}]$, the reduced-order model $\hat{\Omega}$ is asymptotically stable in the absence of input ($u = 0$);
- the error norm $|y(t) - \hat{y}(t)|$, with $\hat{y}(t)$ as the output of $\hat{\Omega}$, is small in some sense, and the corresponding error system satisfies an a priori computable error bound;
- the approximation procedure entails solving only algebraic equations.

3 Characterization of observability and controllability functionals

In this section, we introduce observability and controllability energy functionals for the time delay system (1), and then provide computable functionals that upper/lower bound those energy functionals. These energy functionals give some measure of observability and controllability of the system and their bounds are used for the purpose of model order reduction by truncation.

Before presenting the results of this section, we present a technical lemma (see, e.g., (Curtain and Zwart, 1995)).

Lemma 1. *Consider a system of the form* (1). *If* $x_{t_0} \in \mathcal{W}_n$ *for* $t_0 \in \mathbb{R}$ *and* $u \in \mathcal{L}_\infty([t_0, t_1], \mathbb{R}^m)$ *for* $t_1 \geq t_0$, *then* $x_t \in \mathcal{W}_n$ *for* $t \in [t_0, t_1]$.

3.1 Observability functional

The observability energy functional of a system characterizes the output energy of that system for a non-zero initial condition and zero input. A formal definition is given below (Besselink et al., 2017).

Definition 1. *The observability functional of the system* (1) *is the functional* $L_o : \mathcal{C}_n \to \mathbb{R}$ *defined as*

$$L_o(\varphi) = \int_0^\infty |y(t)|^2 dt, \tag{2}$$

where $y(\cdot)$ *is the output of the system* (1) *for the initial condition* $x_0 = \varphi$ *and zero input.*

We note that the existence of the observability functional in (2) is guaranteed by asymptotic stability of the system Ω for $u = 0$. Computing the observability functional of this system is, however, challenging, if not impossible in general. This motivates the next lemma, that provides a computable delay-dependent functional shown to upper-bound the observability functional of Ω.

Lemma 2. *Consider the asymptotically stable system* (1). *If there exist symmetric matrices* $Q > 0$, $Q_a \geq 0$, *and a scalar* $\alpha_o > 0$ *such that*

$$M_o = \begin{bmatrix} N_{11} & QA_d + \alpha_o Q & C^T & \tau A^T Q \\ * & -\alpha_o Q - Q_a & C_d^T & \tau A_d^T Q \\ * & * & -I_p & 0 \\ * & * & * & -\alpha_o^{-1} Q \end{bmatrix} \leq 0, \tag{3}$$

with $N_{11} = QA + A^T Q - \alpha_o Q + Q_a$, *then the functional* $E_o : \mathcal{W}_n \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ *defined as*

$$E_o(\varphi, \dot{\varphi}) = E_o^1(\varphi) + E_o^2(\dot{\varphi}), \tag{4}$$

with

$$E_o^1(\varphi) = \varphi(0)^T Q \varphi(0) + \int_{-\tau}^0 \varphi^T(s) Q_a \varphi(s)\, ds, \tag{5a}$$

$$E_o^2(\dot{\varphi}) = \alpha_o \tau \int_{-\tau}^0 \int_\theta^0 \dot{\varphi}^T(s) Q \dot{\varphi}(s)\, ds d\theta, \tag{5b}$$

satisfies

$$E_o(\varphi, \dot{\varphi}) \geq L_o(\varphi), \tag{6}$$

for all $\varphi \in \mathcal{W}_n$ *and with* L_o *as in Definition 1.*

Proof. Since $\varphi \in \mathcal{W}_n$ and $u(t) = 0$, $x_t \in \mathcal{W}_n$ for all $t \geq 0$, due to Lemma 1. Consequently $E_o(x_t, \dot{x}_t)$ is well-defined for each $t \geq 0$. We can compute an upper bound for its time-derivative along the trajectories of the system (1) for $u(t) = 0$, $t \geq 0$. For $E_o^1(x_t)$ from (5a), we have

$$\begin{aligned} \dot{E}_o^1(x_t) = {} & \dot{x}^T(t) Q x(t) + x^T(t) Q \dot{x}(t) + x^T(t) Q_a x(t) \\ & - x^T(t-\tau) Q_a x(t-\tau) =: \xi_o^T(t) N_1 \xi_o(t), \end{aligned} \tag{7}$$

where $\xi_o^T(t) := [\ x^T(t) \quad x^T(t-\tau)\ ]$ and

$$N_1 = \begin{bmatrix} QA + A^T Q + Q_a & QA_d \\ * & -Q_a \end{bmatrix}. \tag{8}$$

Next, we compute an upper bound for the time-derivative of $E_o^2(\dot{x}_t)$ in (5b) in terms of ξ_o. From (5b) and the use of the Leibniz integration rule, we obtain

$$\begin{aligned} \dot{E}_o^2(\dot{x}_t) = {} & \alpha_o \tau \int_{-\tau}^0 \dot{x}^T(t) Q \dot{x}(t)\, d\theta \\ & - \alpha_o \tau \int_{-\tau}^0 \dot{x}^T(t+\theta) Q \dot{x}(t+\theta)\, d\theta \\ = {} & \alpha_o \tau^2 \dot{x}^T(t) Q \dot{x}(t) - \alpha_o \tau \int_{-\tau}^0 \dot{x}^T(t+\theta) Q \dot{x}(t+\theta)\, d\theta. \end{aligned} \tag{9}$$

Now, we use Jensen's inequality (Gu, 2000) and the Newton–Leibniz formula to bound the second term in the

right-hand side of (9), resulting in

$$-\alpha_o\tau\int_{t-\tau}^{t}\dot{x}^T(s)Q\dot{x}(s)\,ds \tag{10a}$$
$$\leq -\alpha_o\left(\int_{t-\tau}^{t}\dot{x}^T(s)\,ds\right)Q\left(\int_{t-\tau}^{t}\dot{x}(s)\,ds\right) \tag{10b}$$
$$= -\alpha_o(x(t)-x(t-\tau))^TQ\,(x(t)-x(t-\tau)) \tag{10c}$$
$$=: \xi_o^T(t)N_2\xi_o(t), \tag{10d}$$

where

$$N_2=\alpha_o\begin{bmatrix}-Q & Q\\ * & -Q\end{bmatrix}. \tag{11}$$

Substituting $\dot{x}(t)$ from (1), for $u=0$, into the first term in the right-hand side of (9) yields, for $K=\tau[A\ A_d]$,

$$\alpha_o\tau^2\dot{x}^T(t)Q\dot{x}(t) =: \alpha_o\xi_o^T(t)K^TQK\xi_o(t). \tag{12}$$

The summation of the relations (7), (10d) and (12) leads to an upper bound on the time-derivative of $E_o(x_t,\dot{x}_t)$ along the solution of (1) for $u=0$ as

$$\dot{E}_o(x_t,\dot{x}_t)\leq\xi_o^T(t)\left(\sum_{i=1}^{2}N_i+\alpha_oK^TQK\right)\xi_o(t). \tag{13}$$

Considering that $|y(t)|^2=|[\,C\ C_d\,]\xi_o|^2$, see (1) with $u=0$, one concludes that if

$$\sum_{i=1}^{2}N_i+\alpha_oK^TQK+\begin{bmatrix}C^T\\ C_d^T\end{bmatrix}\begin{bmatrix}C & C_d\end{bmatrix}\leq 0, \tag{14}$$

then $\dot{E}_o(x_t,\dot{x}_t)\leq -|y(t)|^2$. Integration of both sides of this inequality over the interval $[0,T]$ leads to

$$E_o(x_T,\dot{x}_T)-E_o(x_0,\dot{x}_0)\leq -\int_0^T|y(t)|^2dt. \tag{15}$$

In this case, recalling the asymptotic stability of the system for $u=0$, that implies that $\lim_{T\to\infty}E_o(x_T,\dot{x}_T)=0$, and also the fact that $x_0=\varphi$, one obtains

$$E_o(\varphi,\dot{\varphi})\geq L_o(\varphi), \tag{16}$$

as follows from (15) for $T\to\infty$ and Definition 1.

It thus remains to be shown that (3) implies (14). However, using Schur complements, (3) and (14) can be observed to be equivalent. This completes the proof. □

3.2 Controllability functional

A controllability functional characterizes the minimum input energy required by a system, of the form (1), to reach from the zero-state to a final state φ. A formal definition is provided below (Besselink et al., 2017).

Definition 2. *The controllability functional of the system* (1) *is the functional* $L_c:\mathcal{D}_n\to\mathbb{R}$ *defined as*

$$L_c(\varphi)=\inf_u\left\{\int_{-\infty}^{0}|u(t)|^2\,dt\;\middle|\;u\in\mathcal{L}_2\cap\mathcal{L}_\infty\big((-\infty,0],\ \mathbb{R}^m\big),\ \lim_{T\to\infty}x_{-T}=0,\ x_0=\varphi\right\}, \tag{17}$$

where x_t *is the solution of* (1) *for* $u(\cdot)$ *that satisfies the above and* $\mathcal{D}_n\subset\mathcal{C}_n$ *is the domain of* L_c*, that is the space of function segments* φ *for which* L_c *is well-defined.*

Remark 1. *We note that this definition is stated regardless of the stability properties of the system.*

The following lemma provides a computable lower-bound on the controllability energy functional.

Lemma 3. *Consider the system* (1)*. If there exists symmetric matrices* $P>0$*,* $P_a\geq 0$*, and a scalar* $\alpha_c>0$ *such that*

$$M_c=\begin{bmatrix}M_{11} & A_dP+\alpha_cP & B & \tau PA^T\\ * & -\alpha_cP-P_a & 0 & \tau PA_d^T\\ * & * & -I_m & \tau B^T\\ * & * & * & -\alpha_c^{-1}P\end{bmatrix}\leq 0, \tag{18}$$

with $M_{11}=AP+PA^T-\alpha_cP+P_a$*, then the functional* $E_c:\mathcal{W}_n\times\mathcal{L}_2([-\tau,0],\mathbb{R}^n)\to\mathbb{R}$ *defined as*

$$E_c(\varphi,\dot{\varphi})=E_c^1(\varphi)+E_c^2(\dot{\varphi}), \tag{19}$$

with

$$E_c^1(\varphi)=\varphi^T(0)R\varphi(0)+\int_{-\tau}^{0}\varphi^T(s)R_a\varphi(s)\,ds, \tag{20a}$$
$$E_c^2(\dot{\varphi})=\alpha_c\tau\int_{-\tau}^{0}\int_{\theta}^{0}\dot{\varphi}^T(s)R\dot{\varphi}(s)\,dsd\theta, \tag{20b}$$

$R=P^{-1}$ *and* $R_a=P^{-1}P_aP^{-1}$*, satisfies*

$$E_c(\varphi,\dot{\varphi})\leq L_c(\varphi), \tag{21}$$

for all $\varphi\in\mathcal{D}_n\cap\mathcal{W}_n$ *and* L_c *as in Definition 2.*

Proof. We first compute the time-derivative of $E_c(x_t,\dot{x}_t)$ along the trajectory of the system (1) for an initial condition belonging to $\mathcal{W}_n$ and bounded piecewise continuous input $u(\cdot)$, such that $E_c(x_t,\dot{x}_t)$ is well-defined due to Lemma 1. For the first component $E_c^1(x_t)=x^T(t)Rx(t)+\int_{t-\tau}^{t}x^T(s)R_ax(s)\,ds$, we obtain

$$\dot{E}_c^1(x_t)=:\xi_c^T(t)M_1\xi_c(t), \tag{22}$$

where $\xi_c^T(t):=[x^T(t)\ x^T(t-\tau)\ u^T(t)]$ and

$$M_1=\begin{bmatrix}RA+A^TR+R_a & RA_d & RB\\ * & -R_a & 0\\ * & * & 0\end{bmatrix}.$$

Next, we consider $E_c^2(\dot{x}_t)$ in (20b). Noting that $E_c^2(\dot{x}_t) = \alpha_c\tau\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)\,dsd\theta$, we obtain

$$\dot{E}_c^2(\dot{x}_t)=\alpha_c\tau^2\dot{x}^T(t)R\dot{x}(t)-\alpha_c\tau\int_{-\tau}^{0}\dot{x}^T(t+\theta)R\dot{x}(t+\theta)\,d\theta, \tag{23}$$

where the Leibniz integral rule is employed. Now, we aim to bound the right-hand side of (23) by some function in terms of $\xi_c(t)$. To this end, note that

$$-\alpha_c\tau\int_{-\tau}^{0}\dot{x}^T(t+\theta)R\dot{x}(t+\theta)\,d\theta\leq\xi_c^T(t)M_2\xi_c(t), \tag{24}$$

where a change of the integration variable and Jensen's inequality have been used. Moreover, M_2 is given by

$$M_2=\text{blkdiag}\left\{\alpha_c\begin{bmatrix}-R & R\\ * & -R\end{bmatrix},0\right\}.$$

Now, substituting $\dot{x}(t)$ from (1) into the the first term on the right-hand side of (23) yields

$$\alpha_c\tau^2\dot{x}^T(t)R\dot{x}(t)=:\alpha_c\xi_c^T(t)L^TRL\xi_c(t), \tag{25}$$

where $L=\tau[\,A\;A_d\;B\,]$. Then, the summation of the results in (22), (24) and (25) gives an upper bound for the time-derivative of E_c along the trajectories of (1). After adding and subtracting $|u(t)|^2$, we obtain

$$\dot{E}_c(x_t,\dot{x}_t)\leq\xi_c^T(t)\left(\sum_{i=1}^{3}M_i+\alpha_cL^TRL\right)\xi_c(t)+|u(t)|^2, \tag{26}$$

where $M_3=\text{blkdiag}\{0,0,-I_m\}$. Assume that

$$\sum_{i=1}^{3}M_i+\alpha_cL^TRL\leq 0, \tag{27}$$

such that $\dot{E}_c(x_t,\dot{x}_t)\leq|u(t)|^2$ along trajectories of (1). In this case, integration of (26) over $[-T,0]$ yields

$$E(x_0,\dot{x}_0)-E(x_{-T},\dot{x}_{-T})\leq\int_{-T}^{0}|u(t)|^2\,dt, \tag{28}$$

for any $u(\cdot)$ with the aforementioned properties. Now, consider any such input that, additionally, belongs to $\mathcal{L}_2((-\infty,0],\mathbb{R}^m)$ such that the corresponding solution of (1) satisfies $x_0=\varphi\in\mathcal{W}_n$ and also $\lim_{T\to\infty}E_c\left(x_{-T},\dot{x}_{-T}\right)=0$. Then, we obtain

$$E_c(\varphi,\dot{\varphi})\leq\int_{-\infty}^{0}|u(t)|^2\,dt, \tag{29}$$

such that the result (21) holds as a consequence of Definition 2.

To complete the proof, it remains to be shown that the satisfaction of (18) is equivalent to the satisfaction of (27). To this end, we define $R:=P^{-1}$, $R_a:=P^{-1}P_aP^{-1}$. It is observed that pre- and post-multiplication of (18) by $\text{blkdiag}\{R,R,I_m,R\}$ and the application of Schur complements to the results lead to (27). □

Remark 2. *As the solutions to the matrix inequalities in* (3) *and* (18) *are not unique, we may solve those in the presence of appropriate cost functions to form optimization problems involving matrix inequalities. In this way, the solution space of these inequalities can be limited to solutions more suitable for model reduction. In this paper, we choose the cost functions to be* $J_c=\text{trace}(P)$ *and* $J_o=\text{trace}(Q)$*, which is a heuristic to obtain tight bounds on the observability and controllability functionals (Sandberg, 2010).*

Remark 3. *The matrix inequalities in* (3) *and* (18) *are similar to delay-dependent linear matrix inequalities (LMIs) representing sufficient conditions for asymptotic stability of time delay systems of the form* (1) *(see, e.g., (Fridman, 2014, Section 3.6.2)). Similar to LMI conditions for stability, inequalities in* (3) *and* (18) *suffer from some degree of conservatism. In general, the feasibility of those stability LMIs guarantees the existence of solutions to the inequalities* (3) *and* (18)*. Moreover, we can show that for sufficiently small values of* τ*, these inequalities are always solvable.*

Before closing this section, we give the following remark.

Remark 4. *The delay-independent results of (Besselink et al., 2017) can be considered as a special case of the results of the current paper. In particular, by considering* α_o *and* α_c *as free parameters and letting these converge to zero, the inequalities* (3) *and* (18) *become equivalent to their counterparts in (Besselink et al., 2017).*

4 Model order reduction by truncation

We are now in a position to explain how a general model of the form (1) can be reduced through a truncation procedure. Generally, in a truncation procedure, we consider a partitioned form of $x(t)$ and x_t (and φ) as follows:

$$x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},\;x_t=\begin{bmatrix}x_{1,t}\\ x_{2,t}\end{bmatrix},\;\varphi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \tag{30}$$

where $x_1(t)\in\mathbb{R}^k$ and $\varphi_1\in\mathcal{W}_k$, with $1\leq k<n$. The corresponding partitioning of the system matrices is

$$\begin{aligned}&A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\;A_d=\begin{bmatrix}A_{d,11} & A_{d,12}\\ A_{d,21} & A_{d,22}\end{bmatrix},\;B=\begin{bmatrix}B_1\\ B_2\end{bmatrix},\\ &C=\begin{bmatrix}C_1 & C_2\end{bmatrix},\;C_d=\begin{bmatrix}C_{d,1} & C_{d,2}\end{bmatrix}.\end{aligned} \tag{31}$$

Using this partitioning, a reduced-order approximation of (1) is obtained by truncation of the dynamics corresponding to x_2. Such an approximate model reads

$$\hat{\Omega}:\begin{cases}\dot{\zeta}(t)=A_{11}\zeta(t)+A_{d,11}\zeta(t-\tau)+B_1u(t),\\ \hat{y}(t)=C_1\zeta(t)+C_{d,1}\zeta(t-\tau)+Du(t),\\ \zeta_0=\hat{\varphi},\end{cases} \tag{32}$$

where $\zeta(t)\in\mathbb{R}^k$, $\hat{y}(t)\in\mathbb{R}^p$ is an approximate of $y(t)$, and $\hat{\varphi}\in\mathcal{W}_k$ is the initial condition.

The system $\hat{\Omega}$ approximates x_1 in the partitioned coordinate. As can be clearly seen from (32), this model approximation preserves the delay structure. Moreover, if the matrices Q and P satisfying Lemmas 2 and 3 in Section 3 have a particular structure, then the described model order reduction method will enjoy some other important properties. In particular, this will allow us to guarantee stability preservation and compute an a priori bound on the reduction error. As a stepping stone, it is shown that the observability and controllability energy functionals of the reduced system can be characterized in terms of those of the original (high-order) system.

Lemma 4. *Let condition* (3) *hold for a scalar* $\alpha_o > 0$ *and symmetric matrices* $Q > 0$ *and* $Q_a \geq 0$ *of the form*

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \; Q_a = \begin{bmatrix} Q_{a,11} & Q_{a,12} \\ Q_{a,21} & Q_{a,22} \end{bmatrix}, \tag{33}$$

with $Q_1, Q_{a,11} \in \mathbb{R}^{k\times k}$. *Then, the observability functional* $\hat{L}_o : \mathcal{W}_k \to \mathbb{R}$ *of the reduced-order system* (32) *exists, and the functional* $\hat{E}_o : \mathcal{W}_k \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^k) \to \mathbb{R}$ *given as*

$$\hat{E}_o(\hat{\varphi}, \dot{\hat{\varphi}}) = \hat{E}_o^1(\hat{\varphi}) + \hat{E}_o^2(\dot{\hat{\varphi}}), \tag{34}$$

with

$$\hat{E}_o^1(\hat{\varphi}) = \hat{\varphi}^T(0)Q_1\hat{\varphi}(0) + \int_{-\tau}^{0} \hat{\varphi}^T(s)Q_{a,11}\hat{\varphi}(s)\, ds, \tag{35}$$

$$\hat{E}_o^2(\dot{\hat{\varphi}}) = \alpha_o\tau \int_{-\tau}^{0}\int_{\theta}^{0} \dot{\hat{\varphi}}^T(s)Q_1\dot{\hat{\varphi}}(s)\, ds d\theta, \tag{36}$$

satisfies $\hat{E}_o(\hat{\varphi}, \dot{\hat{\varphi}}) \geq \hat{L}_o(\hat{\varphi})$ *for all* $\hat{\varphi} \in \mathcal{W}_k$.

Proof. The matrices Q and Q_a and the scalar α_o are such that (3) holds. Thus, for any matrix Ψ of appropriate dimensions it can be show that

$$\Psi^T M_o \Psi \leq 0, \tag{37}$$

with M_o as in (3). Choosing $\Psi = \text{blkdiag}\{\psi, \psi, I_p, \psi\}$, with $\psi = [I_k \;\; 0_{k\times(n-k)}]^T$, and exploiting the block-diagonal structure of Q, it is straightforward to show that (37) leads to an inequality of the form (3), in terms of Q_1, $Q_{a,11}$ and α_o, for the reduced-order system $\hat{\Omega}$. Given the fact that $\alpha_o^{-1}Q_1 > 0$, this implies that an inequality of the form (15) holds for the reduced-order system. Specifically,

$$\hat{E}_o(\hat{\varphi}, \dot{\hat{\varphi}}) \geq \int_0^T |\hat{y}(t)|^2 dt + \hat{E}_o(\zeta_T, \dot{\zeta}_T), \tag{38}$$

holds for any $T \geq 0$. Given the fact that $\hat{E}_o(\zeta_t, \dot{\zeta}_t) \geq 0$ for all $\zeta_t \in \mathcal{W}_k$, we obtain $\hat{E}_o(\hat{\varphi}, \dot{\hat{\varphi}}) \geq \hat{L}_o(\hat{\varphi})$ for all $\hat{\varphi} \in \mathcal{W}_k$, which is obtained by considering $T \to \infty$ in (38). This result, with the fact that $\hat{E}_o(\hat{\varphi}, \dot{\hat{\varphi}})$ is bounded, further implies the existence of $\hat{L}_o(\hat{\varphi})$ for all $\hat{\varphi} \in \mathcal{W}_k$. □

Analogously, one can use the properties of the controllability functional of the original system to characterize the controllability functional of the reduced system.

Lemma 5. *Let condition* (18) *hold for a scalar* $\alpha_c > 0$ *and symmetric matrices* $P > 0$ *and* $P_a \geq 0$ *of the form*

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad P_a = \begin{bmatrix} P_{a,11} & P_{a,12} \\ P_{a,21} & P_{a,22} \end{bmatrix}, \tag{39}$$

with $P_1, P_{a,11} \in \mathbb{R}^{k\times k}$, *and* $\hat{L}_c(\hat{\varphi}) : \mathcal{D}_k \to \mathbb{R}$ *is the controllability functional of the reduced system* (32). *Then, the functional* $\hat{E}_c : \mathcal{W}_k \times \mathcal{L}_2([-\tau, 0], \mathbb{R}^k) \to \mathbb{R}$ *given as*

$$\hat{E}_c(\hat{\varphi}, \dot{\hat{\varphi}}) = \hat{E}_c^1(\hat{\varphi}) + \hat{E}_c^2(\dot{\hat{\varphi}}), \tag{40}$$

with

$$\hat{E}_c^1(\hat{\varphi}) = \hat{\varphi}^T(0)R_1\hat{\varphi}(0) + \int_{-\tau}^{0} \hat{\varphi}^T(s)R_{a,11}\hat{\varphi}(s)\, ds, \tag{41}$$

$$\hat{E}_c^2(\dot{\hat{\varphi}}) = \alpha_c\tau \int_{-\tau}^{0}\int_{\theta}^{0} \dot{\hat{\varphi}}^T(s)R_1\dot{\hat{\varphi}}(s)\, ds d\theta, \tag{42}$$

$R_1 = P_1^{-1}$ *and* $R_{a,11} = P_1^{-1}P_{a,11}P_1^{-1}$, *satisfies* $\hat{E}_c(\hat{\varphi}, \dot{\hat{\varphi}}) \leq \hat{L}_c(\hat{\varphi})$ *for all* $\hat{\varphi} \in \mathcal{D}_k \cap \mathcal{W}_k$.

Proof. The proof is similar to that of Lemma 4 and is omitted for the sake of brevity. □

Lemmas 4 and 5 imply that the observability and controllability functionals of the reduced-order system can be obtained by relevant parts of the energy functionals of the original system (1) when Q in (5) and P in (20) are block-diagonal as in (33) and (39), respectively.

Next, we define a partially-balanced realization of a time delay system. This will enable us to later state the main properties of the described reduction method.

Definition 3. *A realization as in (1) is said to be partially-balanced if there exists symmetric matrices* $Q > 0$, $Q_a \geq 0$ *and a scalar* $\alpha_o > 0$ *satisfying (3), symmetric matrices* $P > 0$, $P_a \geq 0$ *and a scalar* $\alpha_c > 0$ *satisfying (18), and, additionally,* P *and* Q *are such that*

$$P = Q = \Sigma = \text{blkdiag}\{\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \cdots, \sigma_q I_{m_q}\}. \tag{43}$$

Here, the constants $\sigma_i > 0$, *satisfying* $\sigma_i > \sigma_{i+1}$, $i \in \{1, ..., q-1\}$, *are called singular values and* $\Sigma_{i=1}^q m_i = n$.

The matrices Q and P play a similar role as the Gramians in balanced truncation for finite-dimensional systems. Therefore, it is natural to expect that there exists a coordinate transformation T dependent on Q and P that transforms (1) into a partially-balanced form. The next lemma states this result in a formal manner, which can be proved using standard results in, e.g., Dullerud and Paganini (2010).

Lemma 6. *Let there exists symmetric matrices* $Q > 0$ *and* $Q_a \geq 0$ *and a scalar* $\alpha_o \geq 0$ *satisfying (3), and symmetric matrices* $P > 0$ *and* $P_a \geq 0$ *and a scalar* $\alpha_c > 0$ *satisfying (18). Then, there exists a coordinate transformation* $x(t) = Tz(t)$ *such that the realization in the new coordinates is partially-balanced, i.e., the nonsingular matrix* T *can be chosen such that* $T^TQT = T^{-1}PT^{-T} = \Sigma$, *with* Σ, *as in (43), being the solution (for* Q *and* P *simultaneously) of* (3) *and* (18).

In the literature on finite-dimensional systems, a realization of a system is said to be balanced if 1) the states that are easy to observe are those which are simultaneously easy to control, and vice versa, and, 2) the state components are absolutely ordered in terms of their contribution to the input-output behaviour of the system (Gugercin and Antoulas, 2004). However, the transformed system due to Lemma 6 does not fully fulfil these properties, mainly because the balancing procedure is performed only with respect to $x(t)$ in a finite-dimensional Euclidean space while the state of a time delay system is a function segment, x_t in this case. For this reason, we use the term "partially-balanced realization", rather than "balanced realization". It is also worth noting that since Q and P are dependent on the time delay τ, the transformation T is delay-dependent.

The model reduction method described here preserves not only the delay-structure of the system, but also its stability properties, as stated in the following theorem.

Theorem 1. *Let the system* (1)*, which is asymptotically stable for zero input, be in a partially-balanced realization and consider the reduced-order system* (32) *obtained by truncation for k such that $k = \Sigma_{i=1}^{r} m_i$ for some $r > 0$ and m_i as in Definition 3. Then, the reduced-order system $\hat{\Omega}$ is asymptotically stable for zero input.*

Proof. The proof can be found in Appendix A. □

Remark 5. *Choosing the reduction order k as in Theorem 1 ensures that $\Sigma_1 \in \mathbb{R}^{k\times k}$ and $\Sigma_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, respectively the upper-left and lower-right blocks of Σ, have no singular values in common. If Σ_1 and Σ_2 have common singular values, it cannot be guaranteed that the reduced-order system is asymptotically stable, i.e., that the state trajectory of reduced system converges to zero for zero input. For an example of such a case, for delay-free systems, see (Pernebo and Silverman, 1982). However, whether or not Σ_1 and Σ_2 have common singular values, the convergence of the output of $\hat{\Omega}$ to zero for zero input is still guaranteed, as a consequence of Lemma 4.*

As stated in the next theorem, an interesting property of the proposed delay-dependent model order reduction method is the availability of a guaranteed and a priori error bound, reflecting the accuracy of approximation.

Theorem 2. *Let the asymptotically stable system Ω, as in* (1)*, be in a partially-balanced realization, as defined in Definition 3, and consider the reduced-order system $\hat{\Omega}$, as in* (32)*, obtained by truncation for $k = \Sigma_{i=1}^{r} m_i$ for some $r > 0$. Moreover, let $\alpha_c = \alpha_o = \alpha$. Then, for any common input function $u \in \mathcal{L}_2([0,T],\mathbb{R}^m) \cap \mathcal{L}_\infty([0,T],\mathbb{R}^m)$ and initial conditions $\varphi = 0$ and $\hat{\varphi} = 0$ for* (1) *and* (32)*, respectively,*

$$\int_0^T |y(t) - \hat{y}(t)|^2 \, dt \leq \varepsilon^2 \int_0^T |u(t)|^2 \, dt,$$

for all $T \geq 0$ and where the error bound ε is given as

$$\varepsilon = 2 \sum_{i=r+1}^{q} \sigma_i, \tag{44}$$

with σ_i as in (43)*.*

Proof. This can be proved by extending the proof of Theorem 7 in (Besselink et al., 2017) to the delay-dependent case. Details are omitted for brevity. □

Remark 6. *Two factors contribute to the error bound ε in* (44)*: the solution to the matrix inequalities in (3) and (18), and, most importantly, the reduction order k. Equation* (44) *assures that a larger k, if designed as in Theorem 1, results in a smaller error bound ε. Moreover, the cost functions mentioned in Remark 2 are just heuristics and, thus, it may be possible to obtain a smaller ε for a given k by choosing a different cost function.*

Remark 7. *Compared to its delay-independent counterpart in (Besselink et al., 2017), the delay-dependent model order reduction method in this paper presents an improvement in two aspects. Firstly, for small delays the new method provides tighter error bounds and more accurate reduced-order models. This stems from the fact that the space of feasible P and Q in this method is larger and less conservative compared to the delay-independent method, especially for small delays, leading to tighter bounds on the observability and controllability functionals. Secondly, the class of systems that can be reduced is extended by relaxing conditions that were necessary for the delay-independent method to be feasible. For instance, one such condition was $A - A_d$ being Hurwitz, which is usually not the case when the delay occurs in the feedback channel of a closed-loop system. By contrast, it is no more a necessary feasibility condition for our method.*

5 Illustrative examples

This section illustrates the results via examples. The proposed model order reduction method is compared with the delay-independent method in (Besselink et al., 2017) and a decomposition method in (van de Wouw et al., 2015). All the involved matrix inequalities are solved by the LMI solver YALMIP (Löfberg, 2004).

Example 1. We consider a system of the form (1) described by

$$A = \begin{bmatrix} -0.91 & -0.62 & 1.61 & 0.06 & 0.27 & 0.38 \\ 0.11 & -0.18 & -0.51 & 0.04 & 0.02 & -0.08 \\ 0.05 & 0.03 & -0.18 & 0.02 & 0.06 & 0.17 \\ 0.02 & 0.29 & 1.63 & -0.80 & -0.16 & 0.05 \\ -0.10 & -0.21 & 0.01 & 0.14 & -0.11 & 0.25 \\ -0.03 & 0.46 & -0.49 & -0.03 & -0.12 & -1.11 \end{bmatrix}, B = \begin{bmatrix} 0.61 \\ -0.11 \\ 0.14 \\ 0.31 \\ 0.13 \\ -0.27 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0.74 & 0.66 & -0.34 & -0.21 & -0.21 & 0.23 \\ -0.14 & -0.26 & 0.24 & 0.11 & 0.21 & 0.07 \\ 0.09 & 0.04 & -0.37 & 0.05 & -0.01 & -0.06 \\ -0.35 & 0.01 & -1.01 & -0.38 & -0.71 & -0.65 \\ 0.39 & 0.20 & -0.12 & 0 & -0.08 & 0.15 \\ -0.75 & -0.33 & 1.26 & 0.07 & 0.40 & 0.01 \end{bmatrix}, C = \begin{bmatrix} 3.2 \\ -1 \\ 29.5 \\ 2 \\ 8.4 \\ 8.5 \end{bmatrix}^T,$$

$$C_d = \begin{bmatrix} -4.5 & -38.2 & -6.5 & -5.6 & 1.7 & 2 \end{bmatrix}, D = 0.3\,.$$

In Fig. 1, the left side, the singular values obtained for the delay-dependent and delay-independent methods are compared for $\tau = 1.6$ s. It is observed that the former gives smaller singular values than the latter. Considering the singular values from the delay-independent method, a fairly sharp decay is observed from the second singular value to the third. Thus, we may approximate the system with a second-order model ($k = 2$) and still expect a good approximation. For the delay-dependent

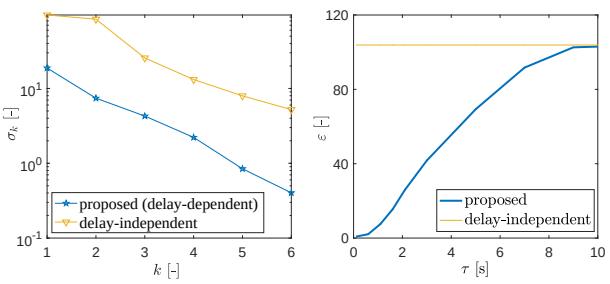

Fig. 1. Comparison of the delay-dependent and delay-independent methods: (left) singular values for $\tau = 1.6$ s, (right) error bounds as a function of τ for the reduction order $k = 2$.

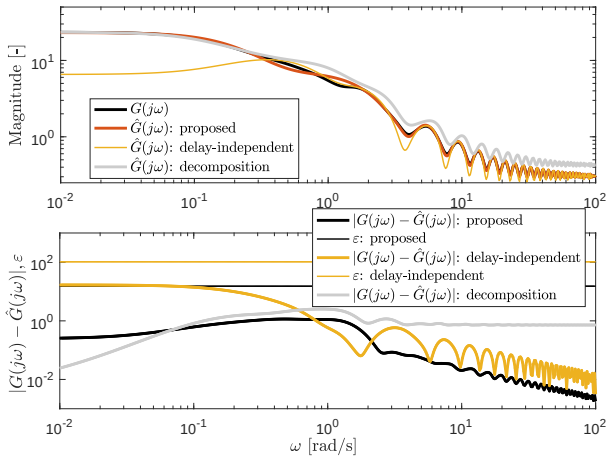

Fig. 2. Comparison of the delay-dependent method with the delay-independent and decomposition methods: (top) magnitude of the frequency response functions, (bottom) error bounds and the magnitude of the error systems for $k = 2$.

approach, a much better accuracy can already be expected at lower orders, due to the smaller singular values. Next, the error bounds are compared. As seen from the right plot in Fig. 1, for $k = 2$, the delay-dependent method gives much tighter error bounds ε for small delays. As expected (see Remarks 4 and 7), as the delay increases, the error bound from the proposed method converges to that of the delay-independent method. In Fig. 2, the frequency response function (from input u to output y) of the original model $G(j\omega)$ is compared to those of the reduced-order models, indicated by $\hat{G}(j\omega)$, obtained from the delay-independent, proposed delay-dependent and the decomposition methods, for $\tau = 1.6$ s and $k = 2$. Clearly, the approximation from the delay-dependent method is more accurate in terms of the $\mathcal{H}_\infty$-norm of the error system $G(j\omega) - \hat{G}(j\omega)$ than for the other methods. Moreover, the stability of the reduced model from the decomposition method is not guaranteed and no a priori error bound is given by this method as the required small-gain condition does not hold. The lower accuracy of this method is because A_d is full-rank.

Example 2. This example illustrates another benefit of the proposed method: the extension of the class of systems that can be reduced. To this end, we consider the model reduction problem of a controlled platoon of eight vehicles from (Scarciotti and Astolfi, 2014), yielding a model of the form (1) with $\tau = 5$ ms and $n = 23$. This model *can not* be reduced by the delay-independent method because $A - A_d$ is not Hurwitz. Alternatively,

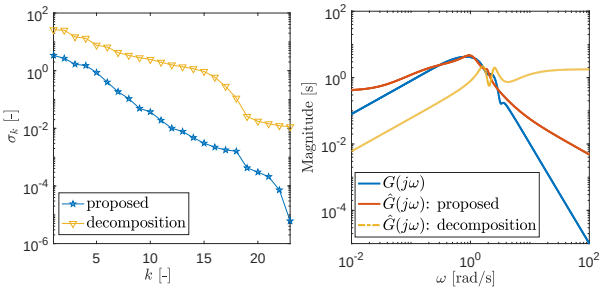

Fig. 3. Comparison of the delay-dependent and decomposition methods: (left) singular values, (right) magnitude of the frequency response functions for $k = 5$.

we use the delay-dependent method. The singular values obtained for the proposed method compared to those for the decomposition method are plotted on the left side of Fig. 3. A comparison between the frequency response function $G(j\omega)$ of the original model with those, indicated by $\hat{G}(j\omega)$, of the reduced models for $k = 5$ is provided on the right side of Fig. 3. Clearly, the proposed method gives a more accurate model approximation. Moreover, the error bound $\varepsilon = 1.66$ obtained by the proposed method is far smaller than $\varepsilon = 93.55$ by the decomposition method. Further, it turns out that in terms of preserving the steady-state response, the method in (Scarciotti and Astolfi, 2014) yields a better model approximation for $k = 5$, whereas the proposed method is superior in terms of the $\mathcal{H}_\infty$-norm of the error system.

6 Conclusions

We presented a balancing-type model order reduction approach for time delay systems based on delay-dependent functionals and matrix inequalities that provide a characterization of observability and controllability properties of the system. The solutions to the matrix inequalities are used as a basis to transform the system into a partially-balanced form, where the system states are sorted in order of their relative contribution to the input-output behaviour of the system. This approach allows for reducing the system by truncating the states with the smallest contribution, while not only preserving stability properties and the delay structure of the original system but providing an a priori computable bound on the reduction error. The effectiveness of the proposed method and the benefits of the delay-dependent nature of the approach have been illustrated through an illustrative example.

Appendix A. Proof of Theorem 1

We prove this theorem by showing that the reduced model has no poles in the closed right-half complex plane, using the counterparts of the inequalities in (3) and (18) for the reduced-order system $\hat{\Omega}$ in (32). Let $\lambda \in \mathbb{C}$ be a root of the characteristic equation of $\hat{\Omega}$, i.e.,

$$\det\left(\lambda I_k - \hat{A}(\lambda)\right) = 0, \tag{45}$$

with $\hat{A}(\lambda) = A_{11} + A_{d,11}e^{-\tau\lambda}$, and let $V \in \mathbb{R}^{k\times d}$ be a matrix such that

$$\left(\lambda I_k - \hat{A}(\lambda)\right) V = 0. \tag{46}$$

Here, d is the geometric multiplicity of λ. Next, Lemma 4 implies that the reduced system fulfils the inequality (consider the counterpart of (14) for the reduced system)

$$\begin{bmatrix} \Sigma_1 A_{11} + A_{11}^T \Sigma_1 + Q_{a,11} & \Sigma_1 A_{d,11} \\ * & -Q_{a,11} \end{bmatrix} + \begin{bmatrix} C_1^T \\ C_{d,1}^T \end{bmatrix} \begin{bmatrix} C_1 & C_{d,1} \end{bmatrix} + \alpha_o \begin{bmatrix} -\Sigma_1 & \Sigma_1 \\ * & -\Sigma_1 \end{bmatrix} + \alpha_o \tau^2 \begin{bmatrix} A_{11}^T \\ A_{d,11}^T \end{bmatrix} \Sigma_1 \begin{bmatrix} A_{11} & A_{d,11} \end{bmatrix} \leq 0,$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is the upper-left block of Σ as in (43). Left and right multiplication of the above inequality by $[I_k \; I_k e^{-\tau\lambda^H}]$ and $[I_k \; I_k e^{-\tau\lambda^H}]^H$, and considering $\lambda = \mu + j\omega$, with $j = \sqrt{-1}$, we obtain

$$\begin{aligned} &\hat{A}^H(\lambda)\Sigma_1 + \Sigma_1 \hat{A}(\lambda) + Q_{a,11}\left(1 - e^{-2\mu\tau}\right) \\ &+ \begin{bmatrix} I_k & I_k e^{-\tau\lambda^H} \end{bmatrix} \begin{bmatrix} C_1^T \\ C_{d,1}^T \end{bmatrix} \begin{bmatrix} C_1 & C_{d,1} \end{bmatrix} \begin{bmatrix} I_k \\ I_k e^{-\tau\lambda} \end{bmatrix} \\ &+ \alpha_o \Sigma_1 \left(e^{-\tau\lambda^H} + e^{-\tau\lambda} - 1 - e^{-2\mu\tau} \right) \\ &+ \alpha_o \tau^2 \hat{A}^H(\lambda)\Sigma_1 \hat{A}(\lambda) \leq 0. \end{aligned} \tag{47}$$

Next, multiplying this result with V^H, from the left, and V, from the right, and using $\hat{A}(\lambda)V = \lambda V$ (see (46)), along with the fact that the forth term in the left-hand side of (47) is always non-negative, yields

$$\begin{aligned} 2\mu V^H \Sigma_1 V + \left(1 - e^{-2\mu\tau}\right) V^H Q_{a,11} V \\ + \alpha_o f\left(\bar{\omega}, \mu\right) V^H \Sigma_1 V \leq 0, \end{aligned} \tag{48}$$

where $f(\bar{\omega}, \mu) = 2e^{-\tau\mu}\cos(\bar{\omega}) - 1 - e^{-2\mu\tau} + \tau^2\mu^2 + \bar{\omega}^2$, $\bar{\omega} = \tau\omega$, which is obtained by using $e^{-\tau\lambda^H} + e^{-\tau\lambda} = 2e^{-\tau\mu}\cos(\omega\tau)$ and $\lambda^H\lambda = \mu^2 + \omega^2$. It can be shown that, except for the origin (i.e., $\lambda = 0$), when $\tau > 0$, and for the imaginary axis (i.e., $\lambda = j\omega$, $\omega \in \mathbb{R}$), when $\tau = 0$, the characteristic equation (45) cannot have any of its roots in the closed right-half plane, since such roots cannot satisfy (48).

To complete the proof (considering $\tau > 0$, as a similar procedure can be followed for $\tau = 0$), it has to be shown that there are no roots of the reduced system at the origin, i.e., $\lambda = 0$ is not a root of (45). Here, we complete the proof by contradiction, i.e., it is initially assumed that $\lambda = 0$ is a root of (45), and then it is shown that this assumption leads to results that contradict the asymptotic stability of the original system (1). Now, for $\lambda = 0$, (47) results in

$$\begin{aligned} \hat{A}^T(0)\Sigma_1 + \Sigma_1 \hat{A}(0) + (C_1 + C_{d,1})^T (C_1 + C_{d,1}) \\ + \alpha_o \tau^2 \hat{A}^T(0)\Sigma_1 \hat{A}(0) \leq 0. \end{aligned} \tag{49}$$

Given that $\alpha_o\tau^2\hat{A}^T(0)\Sigma_1\hat{A}(0) \geq 0$, there exists a matrix $\tilde{C}_1$ with appropriate dimensions such that $\hat{A}^T(0)\Sigma_1 + \Sigma_1\hat{A}(0) + (C_1 + C_{d,1})^T (C_1 + C_{d,1}) + \tilde{C}_1^T\tilde{C}_1 = 0$. Now, left and right multiplication of V^H and V by this result and using the fact that $\lambda = 0$, we obtain

$$(C_1 + C_{d,1})V = 0, \quad \tilde{C}_1 V = 0. \tag{50}$$

Moreover, multiplication from the right of (49) with V and using (50), yields

$$\hat{A}^T(0)\Sigma_1 V + \lambda\Sigma_1 V = 0. \tag{51}$$

Next, we multiply the inequality (18) from the left with $[I_{2n+m} \; 0_{(2n+m)\times n}]$ and from the right with its transpose to obtain

$$\begin{bmatrix} A\Sigma + \Sigma A^T - \alpha_c\Sigma + P_a & A_d\Sigma + \alpha_c\Sigma & B \\ * & -\alpha_c\Sigma - P_a & 0 \\ * & * & -I_m \end{bmatrix} \leq 0. \tag{52}$$

Using the Schur complement on the above, and then multiplying the resulting inequality with $[I_n \; I_n]$ from the left and with its transpose from the right yields

$$\Sigma(A + A_d)^T + (A + A_d)\Sigma + BB^T \leq 0. \tag{53}$$

Therefore, there must exist a matrix $\tilde{B}$ for which

$$\Sigma(A + A_d)^T + (A + A_d)\Sigma + BB^T + \tilde{B}\tilde{B}^T = 0. \tag{54}$$

Likewise, (3) implies there exists a matrix $\tilde{C}$ such that

$$\begin{aligned} (A + A_d)^T\Sigma + \Sigma(A + A_d) + (C + C_d)^T(C + C_d) \\ + \tilde{C}^T\tilde{C} = 0. \end{aligned} \tag{55}$$

Now, consider the partitioning in (31) and partition $\tilde{B}$ and $\tilde{C}$ similarly as $\tilde{B}^T = [\tilde{B}_1^T \; \tilde{B}_2^T]$ and $\tilde{C} = [\tilde{C}_1 \; \tilde{C}_2]$. Then, the upper-left blocks of (54) can be written as

$$\Sigma_1\hat{A}^T(0) + \hat{A}(0)\Sigma_1 + B_1B_1^T + \tilde{B}_1\tilde{B}_1^T = 0. \tag{56}$$

Multiplication of (56) from the left with $V^H\Sigma_1$ and from the right with $\Sigma_1 V$, hereby exploiting (51), leads to

$$B_1^T\Sigma_1 V = 0, \quad \tilde{B}_1^T\Sigma_1 V = 0. \tag{57}$$

Next, if (56) is multiplied from the right with $\Sigma_1 V$, one immediately concludes that $\hat{A}(0)\Sigma_1^2 V = \lambda\Sigma_1^2 V$, where the results in (51) and (57) are used. This fact, along with (46) for $\lambda = 0$, implies that $\mathrm{im}(\Sigma_1^2 V) \subset \mathrm{im}V$, leading to the conclusion that there exists an eigenvector of Σ_1^2 in $\mathrm{im}V$, i.e. $\Sigma_1^2 v = \mu^2 v$, with $v \in \mathrm{im}V$ and μ^2 the corresponding eigenvalue. Considering the definition of Σ_1, it is noted that μ is one of the singular values σ_1 to σ_r in (43). Next, multiplication of the lower-left block of (55) with v from the right and that of (54) with $\Sigma_1 v$ from the same side, and then using (50) and (57) results in

$$(A_{12} + A_{d,12})^T\Sigma_1 v + \Sigma_2(A_{21} + A_{d,21})v = 0, \tag{58}$$

$$(A_{21} + A_{d,21})\Sigma_1^2 v + \Sigma_2(A_{12} + A_{d,12})^T\Sigma_1 v = 0. \tag{59}$$

After multiplying (58) from the left with Σ_2, a comparison of the results with (59) leads to

$$\Sigma_2^2(A_{21} + A_{d,21})v = \mu^2(A_{21} + A_{d,21})v, \tag{60}$$

This implies that μ^2 is an eigenvalue of Σ_2^2, while the choice of the reduction order k according to the multiplicities of the parameters σ_i in (43) ensures that the values on the diagonal of Σ_2 are distinct from μ. As a result, (60) implies that $(A_{21} + A_{d,21})\, v = 0$. Now, from this last result and (46) it is concluded that for $\lambda = 0$,

$$\left(\lambda I - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{d,11} & A_{d,12} \\ A_{d,21} & A_{d,22} \end{bmatrix} e^{-\lambda\tau}\right) \begin{bmatrix} v \\ 0 \end{bmatrix} = 0.$$

This result implies that the original system (1) has a pole at zero. This, however, contradicts the fact that the original system is asymptotically stable. Therefore, the assumption that the reduced-order system has a pole at zero is not valid, which, together with the previous results, implies that the poles of the reduced system all have negative real parts and hence $\hat{\Omega}$ is asymptotically stable, and this completes the proof.

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