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# Robust Invariant Sets for Systems Affected by State-Dependent Disturbance

Christopher Townsend<sup>1</sup>, Sorin Olaru<sup>1</sup>, Nikolaos Athanasopoulos<sup>2</sup> and Eleftherios Vlahakis<sup>3</sup>

**Abstract**—We consider the invariant sets of linear systems subject to state-dependent disturbances. By recursively applying set-valued maps on an equivalent extended dynamic representation of the state-disturbance system, we characterise the Robust Positively Invariant (RPI) sets of stable systems subject to state-dependent locally bounded disturbances. For these systems, the minimal RPI (mRPI) set does not necessarily exist. However, when it does, we prove that it is found in the family of closed fixed points of the set-valued maps. This allows the mRPI to be characterised by a much smaller collection of sets compared to the wider collection of all closed RPI sets.

## I. INTRODUCTION

Invariant sets have received much interest in constrained and robust control [3, 9] and set-valued analysis of dynamical systems [2]. For autonomous contractive discrete-time linear systems subject to a bounded additive disturbance, the set of states reachable from the origin under this disturbance is bounded and unique [7, 11]. This set of states corresponds to the minimal Robust Positively Invariant (mRPI) set [8, 13] – also referred to as the 0-reachable set [5]. The theoretical properties and numerical approximations of mRPI sets of systems subject to state-independent disturbances are well understood. Systems subject to state-dependent disturbances have been studied in [6, 10] which focused on the characterisation of ultimate bounds for linear and non-linear systems, in [4, 14] which focused on limitations of constrained systems and in [12] which considered the reachability of such sets. However, the literature is not as mature as in the state-independent case.

We contribute to the understanding of linear systems with state-dependent disturbances by developing a lift-and-project set-valued map which describes the dynamics of the system. We relate the invariant sets – in particular the mRPI set – of the system to the fixed points of this set-valued map. This enables the mRPI set to be constructed via set iterates. We also provide conditions for the existence of the mRPI set for such systems. Excepting local boundedness, we impose no particular structural properties on the state-dependent disturbances. In [1], the disturbance is assumed to be convex and bounded by polyhedra. These assumptions allow [1] to construct RPI sets using parameterised polyhedra.

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## II. PRELIMINARIES

### A. Dynamics and disturbance characterisation

We consider an exponentially stable discrete-time linear system subject to an additive disturbance

$$x(k+1) = Ax(k) + w(k) \quad (1)$$

where  $x(k) \in \mathbb{R}^n$  is the state vector at the time  $k \in \mathbb{N}$ ,  $w(k) \in \mathbb{R}^n$  is an additive disturbance and  $A \in \mathbb{R}^{n \times n}$  is a Schur matrix. We focus on disturbances  $w$  which are subject to a state-dependent bound. Formally, we define the map

$$W : \mathbb{R}^n \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n) \quad (2)$$

where  $\mathcal{P}_{cl}(\mathbb{R}^n)$  is the set of all closed subsets of  $\mathbb{R}^n$ , including the empty set, and consider  $w(k) \in W(x(k))$ . Note that  $W(x(k))$  may be unbounded for each  $x(k)$ . To maintain boundedness of the state  $x(k)$  we require Assumption 1.

**Assumption 1.** For all  $x \in \mathbb{R}^n$  that

$$\hat{w}(x) := \max \{ \|w\| : w \in W(x) \} < \infty$$

and that the disturbance is locally bounded i.e. there is  $M > 0$  such that  $\hat{w}(x) \leq M \|x\|$  for all  $x \in \mathbb{R}^n$ .

Assumption 1 does not imply the boundedness of the disturbance. For some  $x$ , the set  $W(x)$  in (2) may be empty. In this case, the dynamics of (1) are disturbance-free i.e. (1) becomes  $x(k+1) = Ax(k)$ . This is equivalent to setting  $w(k) = 0$  in (1). Hence, when  $W(x) = \emptyset$  we consider the extension  $\overline{W} : \mathbb{R}^n \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n)$

$$\overline{W}(x) := \begin{cases} W(x), & x \in \text{supp}(W) \\ \{0\}, & x \notin \text{supp}(W) \end{cases} \quad (3)$$

where  $\text{supp}(\cdot)$  is the support of the function  $W$ . This extended disturbance, (3), ensures that (1) is forward complete for all  $x$  without affecting the dynamics of the system. We are interested in the *Robust Positive Invariant* (RPI) sets of (1).

**Definition 2** (RPI set). A non-empty set  $\Omega \subset \mathbb{R}^n$  is RPI with respect to (1) if  $Ax + w \in \Omega$  for all  $x \in \Omega$  and all  $w \in \overline{W}(x)$ .

We note a RPI set may not be closed nor bounded. Throughout we focus on bounded RPI sets. The existence of which can be guaranteed under limited restrictions on  $W(\cdot)$  as the disturbance-free dynamics are stable [10]. However, we do not explore these conditions and instead assume the existence of at least one bounded invariant set.

**Assumption 3.** At least one bounded invariant set exists for the system (1).

Within the family of closed invariant sets of (1), we are particularly interested in the *minimal RPI* (mRPI) sets, see Definition 4.

**Definition 4** (mRPI set). The mRPI set *with respect to the disturbance* (3) is the closed RPI set, which is contained in any closed RPI set of (1).

Definition 4 is the same as given in [13, 15]. It should be noted that, unlike in the case of state-independent disturbances where the mRPI set exists and inherits the properties of the disturbance set [7], the existence of an mRPI set is not guaranteed for bounded state-dependent disturbances  $W$  satisfying Assumption 3. However, as proven in Lemma 5, when the mRPI set exists, it may still be characterised as the intersection of all invariant sets.

**Lemma 5.** Let  $\Omega_1, \Omega_2 \subset \mathbb{R}^n$  be two RPI sets under (1). If  $\Omega = \Omega_1 \cap \Omega_2$  is non-empty then  $\Omega$  is a RPI set.

*Proof:* Let  $x \in \Omega_1 \cap \Omega_2$ . As both these sets are invariant  $Ax + w \in \Omega_1$  and  $Ax + w \in \Omega_2, \forall w \in W(x)$  Therefore  $Ax + w \in \Omega_1 \cap \Omega_2$  i.e. the intersection is invariant. ■

**Theorem 6.** A dynamic system (1) admits a mRPI set if and only if any two closed RPI sets  $\Omega_1, \Omega_2 \subset \mathbb{R}^n$  have a non-empty intersection i.e.  $\Omega_1 \cap \Omega_2 \neq \emptyset$ .

*Proof:* If the mRPI set exists then it represents the intersection of all the RPI sets. If, on the contrary, two invariant sets exist and their intersection is empty then there is no set to satisfy Definition 4. ■

Theorem 6 suggests that the existence of a mRPI set is non-trivial for disturbances with state-dependent bounds. We give Example 7 in which the intersection of certain pairs of closed RPI sets is empty.

**Example 7.** Consider the system with dynamics

$$x(k+1) = 0.5x(k) + w(k) \quad (4)$$

with the state-dependent bounds  $w(k) \in W(x)$ , where

$$W(x) = \left\{ w \in \mathbb{R} : \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} w \leq \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The interval  $\Omega_1 = [0, 2]$  is a bounded RPI. However the sets  $\Omega_2 = \{0\}$  and  $\Omega_3 = \{2\}$  are also RPI. Thus the mRPI which may be found from the intersection of all RPI is empty as  $\Omega_2 \cap \Omega_3 = \emptyset$ . Hence no mRPI set exists for this system.

### B. Extended Dynamics

As the disturbance varies as a function of the state  $x$  we may define an equivalent system to (1) by embedding the dynamics in  $\mathbb{R}^{2n}$  and recovering the state by projection. We define this extended system of (1) by

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} A & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(k+1) \quad (5)$$

Note the non-causality introduced in (5) is an artifact of the mathematical description and the non-causal  $w(k+1)$  term on the right-hand side does not affect the dynamics of the system which evolves in a subset of the extended state-space  $\mathbb{R}^n \times \mathbb{R}^n$ :

$$\overline{W} = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{2n} : w \in \overline{W}(x) \right\} \quad (6)$$

which collects all the state-disturbance pairs. When considering properties of the set of admissible disturbances, we use the restriction of  $\mathcal{W} = \overline{W} \cap (\text{supp}(W) \times \mathbb{R}^n)$

$$\mathcal{W} = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{2n} : w \in W(x) \right\} \quad (7)$$

Since the disturbance  $w$  is locally bounded for any state  $x$ , then for any bounded set  $X \subset \mathbb{R}^n$ , we have that  $\overline{W} \cap (X \times \mathbb{R}^n)$  is bounded. We note, in general, without constraining  $x$ , the sets  $\mathcal{W}$  and  $\overline{W}$  are not necessarily bounded.

### III. A LIFT-AND-PROJECT SET VALUED MAP

The RPI sets of (1) may be characterised using a set-valued map on the extended state-space (5)

$$F : \mathcal{P}_{cl}(\mathbb{R}^n) \times \mathcal{P}_{cl}(\mathbb{R}^{2n}) \rightarrow \mathcal{P}_{cl}(\mathbb{R}^n)$$

$$F(X, \overline{W}) := cl(F'(X, \overline{W})) \quad (8)$$

where the map  $F'(X, \overline{W}) := [A \ I](\overline{W} \cap (X \times \mathbb{R}^n))$  is the set of all forward images of a set of states under (1) subject to the state-dependent disturbances  $W(x)$ ,  $cl(\cdot)$  is the closure, while the non-square extension,  $[A \ I] \in \mathbb{R}^{n \times 2n}$  of the matrix  $A$  projects the extended space onto the original state-space.

**Remark 8.** We take the closure of  $F'$  as there may exist  $\mathcal{W}$  such that  $F'(X, \overline{W})$  is open despite  $X$  being closed. For example taking  $X_0 := [0, 1]$  with the system and disturbances in Example 16 we have that  $F'(X_0, \overline{W}) = (0, 1]$ .

Lemma 9 establishes the relationship between the set mapping (8) and the RPI sets of the system (1).

**Lemma 9.** A set  $X \subset \mathbb{R}^n$  is a RPI set for (1) if and only if  $F'(X, \overline{W}) \subseteq X$ .

*Proof:* Suppose  $X$  is a RPI set for (1). For each  $x \in X$ , we have that

$$\begin{aligned} \overline{W} \cap (x \times \mathbb{R}^n) &= \left\{ \begin{pmatrix} x \\ w \end{pmatrix} : w \in \overline{W}(x) \right\} \\ F'(\{x\}, \overline{W}) &= \{Ax + w : w \in \overline{W}(x)\} \end{aligned}$$

Taking the union over all  $x \in X$ , we have

$$\begin{aligned} \bigcup_{x \in X} F'(\{x\}, \overline{W}) &= \bigcup_{x \in X} \{Ax + w : w \in \overline{W}(x)\} \\ &= \{Ax + w : x \in X \text{ and } w \in \overline{W}(x)\} \end{aligned}$$

As  $X$  is RPI, all elements of the form  $Ax + w \in X$ . Thus  $F'(X, \overline{W}) = \{Ax + w : x \in X \text{ and } w \in \overline{W}(x)\} \subseteq X$ .

Suppose  $F'(X, \overline{W}) \subseteq X$ . As noted above  $F'(X, \overline{W})$  is the collection of all the forward images of  $X$  under (1). Thus  $X$  is RPI. ■

We note Lemma 9 holds for both maps  $F$  and  $F'$  when the set  $X$  in Lemma 9 is closed. We now explore the relationship between the RPI sets of (1) and the fixed points of (8).

**Lemma 10.** *Suppose  $X$  is RPI. Then the sets  $F^n(X, \overline{W})$  are RPI for all  $n \in \mathbb{N}$ .*

*Proof:* The proof follows by iteratively applying the map  $F$  to both sides of  $F'(X, \overline{W}) \subset X$ . At each step the  $F^{n+1}(X, \overline{W}) \subseteq F^n(X, \overline{W})$  holds. ■

**Corollary 11.** *If it exists, the mRPI set of (1) is a fixed point*

$$F(X, \overline{W}) = X$$

*Proof:* We proceed by contradiction. Suppose that  $X$  is a mRPI set but  $F(X, \overline{W}) \neq X$ . As  $X$  is a closed RPI from Lemma 9, we have that  $F(X, \overline{W}) \subset X$ . By Lemma 10, as  $F^n(X, \overline{W})$  is RPI for all  $n \in \mathbb{N}$

$$F^n(X, \overline{W}) \subseteq \dots \subseteq F(F(X, \overline{W})) \subseteq F(X, \overline{W}) \subseteq X$$

Either  $F^n(X, \overline{W}) = X$  for all  $n$  in which case  $X$  is a fixed point of  $F$  or there exists  $N$  such that  $F^N(X, \overline{W}) \subset X$ . However if such  $N$  exists then this contradicts the minimality of  $X$  as  $F^N(X, \overline{W})$  is also RPI. ■

As the mRPI is obtained by taking the intersection of all RPI sets of the system (1), if it exists, it is unique. As the disturbance is state-dependent, initialising the dynamics of (1) at a different set of initial conditions may result in a different fixed point. Hence even if there exists a fixed point this does not necessarily correspond to the mRPI set of the system. Lemma 12 explores the relationship between the fixed points of  $F$ .

**Lemma 12.** *Let  $X = F(X, \overline{W})$  and  $X' = F(X', \overline{W})$  be fixed points of  $F$ . Then*

- 1) *if  $X \cap X'$  is non-empty then it is a fixed point of  $F$ .*
- 2)  *$X \cup X'$  is a fixed point of  $F$ .*

*Proof:* We first prove that  $X \cap X' \neq \emptyset$  is a fixed point of  $F$ . Take  $x \in X \cap X'$  as  $X$  is RPI we have that  $f(x) \subset X$ , where  $f(x) \in F(X, \overline{W})$ . Similarly  $f(x) \subset X'$ . Thus  $f(x) \in X \cap X'$  i.e.  $X \cap X'$  is RPI. Suppose  $X \cap X' \supset F(X \cap X', \overline{W})$ . As  $F(X \cap X', \overline{W}) = F(X, \overline{W}) \cap F(X', \overline{W})$  we have that  $F(X, \overline{W}) \cap F(X', \overline{W}) \subset X \cap X'$ . Take  $x \in (X \cap X') \setminus (F(X, \overline{W}) \cap F(X', \overline{W}))$  as  $x \in X$  we have that  $x \in F(X, \overline{W})$ . Thus  $x \notin F(X', \overline{W})$  contradicting the RPI of  $F(X', \overline{W})$ . Therefore no such  $x$  exists and  $X \cap X'$  is a fixed point of  $F$ . The union follows by similar argument. ■

Applying Lemma 12, we can obtain the mRPI set by only taking the intersection of all fixed points of  $F$ . This gives a simpler characterisation of the mRPI set as the fixed points of  $F$  form a small subset of the set of all RPI sets. We conclude this section with Theorem 13 which gives practical sufficient conditions ensuring that (1) admits a mRPI set. Theorem 13 does not require  $W$  to be convex or closed and for  $W(x)$  to be continuous with the proof being non-technical. We note, as in [10], it is possible to ensure the existence of a mRPI set with more general assumptions both on the boundedness and the structure of  $W(x)$  than those used here.

**Theorem 13** (Existence of mRPI). *The system (1) with state-dependent bounds on the disturbances admits a mRPI set if  $W$  is bounded and  $0 \in W(x)$  for all  $x \in \text{supp}(W)$ .*

We postpone proving Theorem 13 as it is simplified by subsequent results.

#### IV. CONVERGENCE OF SET SEQUENCES

Lemmas 14 and 15 prove that recursive sequences of sets generated by the map  $F$  converge to a fixed point of  $F$ . Lemma 14 considers when the sequence is initialised with a set containing a fixed point and Lemma 15 considers when the sequence is initialised with a set which is contained in a fixed point of  $F$  i.e. when the sequence converges to a fixed point from ‘outside’ and ‘inside’ a fixed point respectively.

##### A. Converging from Outside

**Lemma 14.** *Let  $X_0$  be a non-empty, compact RPI set and define the sequence of sets  $(X_k)$  by*

$$X_{k+1} := F(X_k, \overline{W}) \quad (9)$$

*Then the limit  $X_\infty := \lim_{k \rightarrow \infty} X_k$  exists and is a fixed point of the map  $F$ .*

*Proof:* As each  $X_k$  is RPI, from Lemma 10 we have that  $X_k \supseteq F(X_k, \overline{W})$ . Furthermore,  $X_k$  is bounded, closed and non-empty for all  $k$ . Therefore

$$\bigcap_{i \leq k} X_i = X_k \supseteq \bigcap_{j \in \mathbb{N}} X_j$$

Hence in the limit there exists

$$X := \lim_{k \rightarrow \infty} X_k = \bigcap_{j \in \mathbb{N}} X_j \neq \emptyset$$

We note that  $cl(F(X, \overline{W})) = X$  as  $X$  is the intersection of closed sets. ■

*Proof of Theorem 13:* By Lemma 14 there exists at least one fixed point of (8). If the intersection of all fixed points of (8) is non-empty, then the mRPI set exists and is equal to this intersection, by Lemma 12. We now prove that this intersection is non-empty. As  $0 \in W(x)$  for all  $x \in \text{supp}(W)$  we have that 0 is an admissible disturbance for all  $x$ . Hence for any  $x \in \mathbb{R}^n$  we have that  $x(k) = A^k x$  is an admissible solution to (1) subject to the state-dependent disturbances. As  $A$  is Schur, this converges to 0. As all fixed points  $X$  of (8) are closed, we have that  $0 \in X$ . Thus the intersection over all fixed points is non-empty as it must contain the origin. ■

In the case presented in Remark 8 we note that the sequence generated by (9) converges asymptotically. We highlight that the closure of  $X_k$  is taken at each  $k$ . This is as, if any  $X_k$  is open, the limit may not be closed and non-empty.

When initializing with an RPI set, Lemma 14 establishes that the set iteration (9) monotonically decreases to a fixed point of the set map (8). The limit will be the mRPI set if such a set exists and there is a unique fixed point. However, this procedure requires a priori knowledge of a bounded

RPI set  $X_0$ . Finding such sets is non-trivial for a general, potentially unbounded, disturbance  $W(x)$ . However, if  $W(x)$  is globally bounded then the state-dependent disturbance is bounded above by a state-independent disturbance and the RPI set for the system subject to this state-independent disturbance is a RPI set for (1) when subject to the state-dependent disturbance.

### B. Convergence from Inside

We establish in Lemma 15 that, by starting with a set contained in a RPI set which does not contain a RPI set, the set-iteration (10) will *monotonically increase* and converge to a fixed point of  $F$ . This condition is more easily verified than the condition of Lemma 12 as, for example, we can ensure that the initial set  $Z_0$  in Lemma 15 does not contain a RPI set by taking  $Z_0 = \{x\}$  to be a singleton which is positively invariant if and only if  $W(x) = \{(I - A)x\}$ .

**Lemma 15.** *Suppose  $Z_0$  is nonempty and contained in a bounded RPI set. Then, the set iteration*

$$Z_{k+1} = F \left( \bigcup_{i \leq k} Z_i, \overline{W} \right) \cup Z_0 \quad (10)$$

*converges to the – potentially open – fixed point,  $Z_\infty \supseteq Z_0$ .*

*Proof:* By construction, the sequence  $(Z_k)$  is monotone increasing i.e. we have that  $Z_k \subseteq Z_{k+1}$  for all  $k$ . Furthermore as  $Z_0 \subseteq X$  a RPI set  $Z_k \subseteq X$  which is bounded. Hence, the limit exists, is bounded and is given by  $Z_\infty := \lim_{k \rightarrow \infty} Z_k = \bigcup_{k \in \mathbb{N}} Z_k \subseteq X$  ■

Should (10) in Lemma 15 generate open sets, the closure of these sets is not necessarily a fixed point. Example 16 describes a system subject to a state-dependent disturbance for which  $Z_\infty$  is open and the closure of which is neither a fixed point of  $F$  nor (10). It also justifies why the union is required in (10).

**Example 16.** *Consider the one-dimensional system (4) subject to the disturbances  $w(k) \in \overline{W}(k)$ , where*

$$\overline{W}(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

*Then, for  $X_0 := \{\frac{1}{2}\}$ , under the iteration (9)  $X_k = \{\frac{1}{2^k}\} \not\subseteq \{\frac{1}{2^{k+1}}\} = X_{k+1}$  for all  $k \geq 1$ . Whereas for the iteration (10) with  $Z_0 := \{\frac{1}{2}\}$*

$$Z_k = \left\{ \frac{1}{2^i} : i \leq k \right\} \subseteq \left\{ \frac{1}{2^i} : i \leq k+1 \right\} = Z_{k+1}$$

*which is a monotonically increasing sequence of nested sets bounded above converging to the open set*

$$Z_\infty := \left\{ \frac{1}{2^{i+1}} : i \in \mathbb{N} \right\}$$

*The closure  $cl(Z_\infty) = Z_\infty \cup \{0\}$  of  $Z_\infty$  is not a fixed point of  $F$  as*

$$F(\{0\}, \overline{W}) = 1 \notin cl(Z_\infty)$$

Lemma 17 characterises when the limit of the sequence of sets generated by (10) is closed.

**Lemma 17.** *If the sequence (10) converges after finitely many iterations, then  $Z_\infty$  is closed. Furthermore,*

$$Z_\infty = Z := \bigcap_{R \in \mathcal{R}} R$$

*where  $\mathcal{R} := \{R \in \mathbb{R}^n : R \supseteq Z_0 \text{ and } R \text{ is RPI w.r.t. (10)}\}$  i.e.  $Z_\infty$  is the smallest invariant set containing  $Z_0$ .*

*Proof:* Closure of  $Z_\infty$  follows as the finite union of closed sets is closed. Suppose  $Z_\infty \neq Z$ . By assumption there exists  $K$  such that  $Z_k = Z_{k+1}$  for all  $k \geq K$  and these sets are closed i.e.  $Z_k$  is RPI for all  $k \geq K$ .

Furthermore  $Z_k \subset X$  as the intersection of RPI sets is RPI and  $Z_0 \subset Z$ . Thus  $Z_k$  is a smaller RPI containing  $Z_0$ . Contradicting the minimality of  $Z$ . ■

The converse of Lemma 17 does not hold in general. We give a counter-example in Example 18.

**Example 18.** *Suppose the system (4) is subject to the disturbances*

$$W(x) := \begin{cases} \{1\}, & x \neq 0 \\ \{0, 1, 2\}, & x = 0 \end{cases}$$

*Taking  $Z_0 = \{0\}$  and using (10), we have that  $Z_k = \{\frac{2^n-1}{2^{n-1}} : n \leq k\} \cup \{0, 2\}$  which converges to the closed set  $Z_\infty = \{\frac{2^n-1}{2^{n-1}} : n \leq k\} \cup \{0, 2\}$  and  $Z_k \subset Z_{k+1}$  for all  $k$ . However there is no  $N$  such that  $Z_N = Z_\infty$ .*

## V. INVARIANCE OF RPI UNDER CLOSURE

As mentioned in the previous section, the set sequence (10) can converge to an open set. The sequence in Example 16 converges to an open set due to the choice of  $Z_0 \not\subseteq \text{supp}(W)$ . For the same Example 16 if we instead choose  $Z_0 = \{0\}$  the limit becomes  $Z_\infty := \{\frac{1}{2^i} : i \in \mathbb{N}\} \cup \{0\}$  which is closed. In Theorem 19 we give sufficient conditions for when the closure of an open limit of (10) is also RPI.

**Theorem 19.** *Let  $Z_\infty$  be a fixed point of (10) which is not closed. Suppose for any sequence  $(x_k)_{k=0}^\infty \subset Z_\infty$ , which converges to  $\hat{x} \in cl(Z_\infty) \setminus Z_\infty$ , there is  $K$  such that*

$$cl \left( \bigcup_{k > K} W(x_k) \right) \supseteq W(\hat{x})$$

*Then, the closure of the fixed point  $cl(Z_\infty)$  is RPI.*

*Proof:* Let  $\hat{x} \in cl(Z_\infty) \setminus Z_\infty$ . Suppose that  $cl(Z_\infty)$  is not RPI i.e. there exists  $w \in W(\hat{x})$  such that  $A\hat{x} + w = y \notin cl(Z_\infty)$ . If  $y \notin cl(Z_\infty)$  there must exist  $\varepsilon > 0$  such that  $B_\varepsilon(y) \cap cl(Z_\infty) = \emptyset$  – where  $B_\varepsilon(y)$  is the open ball centered at  $y$  with radius  $\varepsilon$ . By assumption  $\hat{x}$  is a limit point of a sequence  $(x_k)_{k=1}^\infty \subset Z_\infty$  such that

$$cl \left( \bigcup_{k > K} W(x_k) \right) \supseteq W(\hat{x})$$

In particular this is true for all subsequences of  $(x_k)$ . Therefore for any  $\varepsilon^* > 0$  we may find  $x_k \in Z_\infty$  such that

$\|\hat{x} - x_k\| < \varepsilon^*$  and  $w \in cl(W(x_k))$ . Additionally, for any  $\varepsilon' > 0$  we may select  $w_k \in B_{\varepsilon'}(w) \cap W(x_k)$ . Fix

$$\varepsilon^* < \frac{\varepsilon}{2(\|A\| + 1)}$$

and  $\varepsilon' < \frac{\varepsilon}{2}$ . Consider

$$\begin{aligned} \|Ax_k + w_k - y\| &= \|A(x_k - \hat{x}) + w - w\| \\ &\leq \|A\| \|x_k - \hat{x}\| + \|w_k - w\| \\ &< \|A\| \varepsilon^* + \varepsilon' \end{aligned}$$

Thus  $\|Ax_k + w_k - y\| < \varepsilon$  contradicting  $y \notin cl(Z_\infty)$ . ■

Corollary 20 gives conditions only on the disturbance set  $\mathcal{W}$  so that the closure of an open limit of (10) is a RPI set.

**Corollary 20.** *Suppose  $\mathcal{W}$  is bounded, convex and  $0 \in \overline{\mathcal{W}}(x)$  for all  $x$ . Let  $Z_\infty$  be the limit of (10). Then  $cl(Z_\infty)$  is RPI.*

As the limit of the map (10) may not be closed we give conditions on  $\mathcal{W}$  so that the closure of a RPI set remains RPI. While Lemma 21 and Theorem 22 are stated with respect to the fixed points discussed in Section IV – for example the set generated by (10) – they apply more generally.

**Lemma 21.** *Consider  $Z_\infty$ , defined as the limit in Lemma 15, and suppose this is open. Then  $cl(Z_\infty)$  is a fixed point of (10) if and only if*

$$dZ_\infty \subset F(dZ_\infty, \overline{\mathcal{W}}) \subseteq cl(Z_\infty)$$

where  $dZ_\infty := cl(Z_\infty) \setminus Z_\infty$ .

*Proof:* As  $cl(Z_\infty) = Z_\infty \cup dZ_\infty$  and  $Z_\infty$  is a fixed point of (10) we see that

$$\begin{aligned} cl(Z_\infty) &= F(cl(Z_\infty), \overline{\mathcal{W}}) \cup X_0 \\ &= F((Z_\infty \cup dZ_\infty), \overline{\mathcal{W}}) \cup Z_0 \\ &= F(Z_\infty, \overline{\mathcal{W}}) \cup Z_0 \cup F(dZ_\infty, \overline{\mathcal{W}}) \\ &= Z_\infty \cup F(dZ_\infty, \overline{\mathcal{W}}) \end{aligned}$$

which holds if and only if  $dZ_\infty \subset F(dZ_\infty, \overline{\mathcal{W}}) \subseteq cl(Z_\infty)$ , i.e. only if all points on the boundary of  $X_\infty$  are contained in the image of the boundary. ■

The objective of Theorem 22 is not to give general conditions which guarantee that  $cl(X_\infty)$  is a fixed point of  $F$  but rather give conditions which do not require a priori knowledge of the limit  $X_\infty$  of the recursive sequence of sets. In Theorem 22 we consider continuity of the set-valued function with respect to the Hausdorff metric  $d_H$ .

**Theorem 22.** *Suppose  $W(x)$  is continuous on the support of  $cl(X_\infty)$ . Then  $cl(X_\infty)$  is a fixed point of  $F$ .*

*Proof:* Take  $x \in \partial X_\infty$ . By assumption, for any  $\varepsilon$ , there is  $y \in X_\infty$  such that  $\|Ax + w_x - Ay - w_y\| < \varepsilon$ . As  $Ay + w_y = y' \in X_\infty$ . We have that  $\|Ax + w_x - y'\| < \varepsilon$  which implies  $Ax + w_x \in cl(X_\infty)$ . ■

We note that the RPI set reached by (10), or its closure, may not be mRPI, see Example 7. Indeed it may contain other RPI sets even if  $X_0 = \{x\}$  is a singleton. We further illustrate this in Example 23.

**Example 23.** *Consider the two dimensional system with dynamics (4), where  $x(k) \in \mathbb{R}^2$  and  $w(k) \in \overline{W}(k)$  with*

$$\overline{W}(x) := \begin{cases} \{(0,0), (0,1), (1,0)\}, & x = (0,0) \\ \{(0.75,0)\}, & x = (0.5,0) \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

If  $X_0 = \{(0,0)\}$ . Then the limit of (10) converges to the closed RPI set

$$X_\infty = \{(0, 0.5^i) : i \in \mathbb{N}\} \cup \{(0.5, 0), (1, 0), (0, 0)\}$$

However the set  $Y := \{(0.5, 0), (1, 0)\}$  is also a closed RPI set and  $Y \subset X_\infty$ . Furthermore if  $X_0 = \{(0.5, 0)\}$ . Then the sequence converges in finitely many steps.

## VI. COMPUTATIONAL CONSIDERATIONS

Finding or estimating the mRPI set, when it exists, is computationally difficult as it is typically non-convex. Thus convex approximations of the estimates of the mRPI are used to simplify the problem using the convex hull  $co(\cdot)$ . A computationally feasible counterpart to (10) is

$$Z_{k+1} = cl(co(F(Z_k, \overline{\mathcal{W}}) \cup Z_0)) \quad (11)$$

Lemma 24 discusses the convergence of the sequence (11) to a convex RPI set. We point the reader to [1] for the proof and further properties of the invariant sets for polyhedral  $\mathcal{W}$ .

**Lemma 24.** *If there exists a convex bounded RPI set  $Z$ , then (11) with  $Z_0 \subseteq Z$  converges to a convex bounded RPI set.*

**Corollary 25.** *If  $\text{supp}(\mathcal{W})$  and  $Z_0$  are bounded. Then the iteration (11) converges to a bounded convex RPI set.*

It should be remarked that even for convex  $\mathcal{W}$ , the convex hull of the mRPI set may not be RPI. As such, the RPI sets obtained via (11), may not be a tight approximation of the mRPI as illustrated by the numerical example below.

### Numerical Example

Consider the perturbed stable dynamics

$$x(k+1) = \begin{bmatrix} 0 & 0.833 \\ -0.833 & 0 \end{bmatrix} x(k) + w(k)$$

where  $w(x(k)) := [1 \ 1]^T d(k)$  and

$$[x(k) \ d(k)]^T \in \mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

with

$$\mathcal{D}_1 := co\{\mathcal{V}_1, \mathcal{V}_2\}, \quad \mathcal{D}_2 := co\{0, \mathcal{V}_1\} \oplus \text{lin}\{l, -\mathcal{V}_3\}$$

and

$$\mathcal{D}_3 = co\{0, \mathcal{V}_2\} \oplus \text{lin}\{-l, \mathcal{V}_3\}$$

where  $\text{lin}(\cdot)$  is the set of all positive linear combinations. The vertices  $\mathcal{V}_i$  and  $l$  are

$$\begin{aligned} \mathcal{V}_1 &= \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0.2 \end{pmatrix} \right\}, l = \begin{pmatrix} -1.2 \\ 0 \\ 0.37 \end{pmatrix} \\ \mathcal{V}_2 &= \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \mathcal{V}_3 = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ -1.5 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ 1.5 \end{pmatrix} \right\} \end{aligned}$$

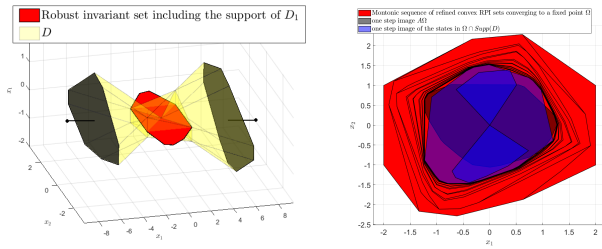


Fig. 1. (Left) RPI set (red) constructed using (11) where  $Z_0 = \text{supp}(\mathcal{D}_1)$ . (Right) Iterative RPI refinement converging to a tight convex RPI set.

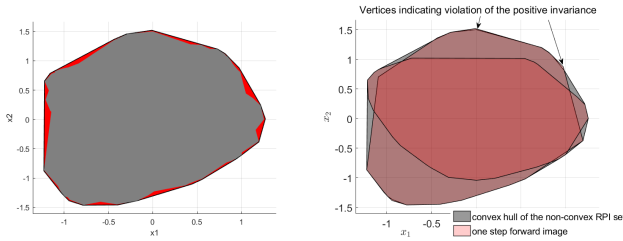


Fig. 2. (Left) mRPI set  $X_\infty$  represented as the union of convex (polyhedral) grey sets, compared with the convex RPI obtained above in red. (Right) Illustration of the violation of positive invariance for  $\text{co}(X_\infty)$ .

This disturbance set is locally bounded but is globally unbounded, as shown by the yellow shaded region in Figure 1. Using (11) an initial RPI is obtained, shown by the red region in Figure 1. Note that the RPI set is bounded despite the unboundedness of the disturbance set  $\mathcal{D}$ . We also note that this RPI set contains regions of the state space which are not in  $\text{supp}(\mathcal{D})$ . If (10) is initialised with the convex RPI obtained from (11) then we obtain a RPI approximation of the non-convex mRPI set. This is shown on the left of Figure 2 and is identical to the one in [1] despite being obtained using a different computational approach.

As mentioned above the convex hull of the mRPI is used to approximate the mRPI set. However in the example presented the convex hull of the mRPI set results in a set which is no longer RPI. This is shown on the right-hand side of the Figure 2. We have taken the convex hull to emphasize in the case of state-dependent disturbances care must be taken whenever additional points are added.

## VII. CONCLUSIONS AND FURTHER WORK

We have considered the invariant sets of systems subject to state-dependent locally bounded disturbances and in particular the minimal invariant set (mRPI) i.e. the closed invariant set contained in all other closed invariant sets. We considered the map  $F$  of all forward images of a set of states subject to the disturbance and two set recursions based on this map. These recursions converge, from the outside and the inside, respectively, to RPI sets, which are fixed points. Furthermore we have proven that the mRPI set – when it exists – is a fixed point of the forward map and the related recursions. Thus to obtain the mRPI set we need only consider the closed fixed points of (9) and (10) rather than the much larger collection of all closed RPI sets.

As demonstrated, the existence of mRPI sets in the case of state-dependent disturbances is a non-trivial problem. Developing less restrictive conditions on  $\mathcal{W}$  which ensure the mRPI exists should assist in finding optimal initial conditions for the recursive maps. Given a mRPI set, we would also like to determine which disturbance sets result in the same mRPI set. The geometric relationship between the maximal disturbance set and the mRPI set is of particular interest.

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