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# Equivalence of state dependent disturbances to piecewise polytopic affine dynamics

Nikolaos Athanasopoulos, Eleftherios Vlahakis, Sorin Olaru, and Christopher Townsend

**Abstract**—We consider linear systems with exogenous signals whose range is constrained within a polytopic set defined in the states-disturbances space. We show they are equivalent to piecewise polytopic affine dynamics defined only in the state space, which is convex on the support of the disturbance set. Given that many interesting setups in control have exogenous signals whose range is coupled with the states, the observations of this note can provide additional insight, and establish alternative, algorithmic procedures for performance analysis, specifically in the retrieval of robust positively invariant sets.

## I. INTRODUCTION

Reachability analysis can address quantitatively important problems related to stability, safety and performance. For linear systems whose exogenous signals, inputs and states are independent from each other, and for polytopic sets, computations involve operations concerning convex hulls, erosions, and projections of polytopic sets on linear spaces, [1]. On the other hand, reachability analysis for systems whose disturbances are input and state dependent is considerably more involved. In fact, even for linear systems and convex state and input constraints, the backward reachable sets can be nonconvex, even non-connected as exposed in [2], with other relevant works in [3]–[8]. Roughly, when the nondeterministic disturbances depend on the state or other signals, one cannot eliminate easily the quantifier corresponding to them, resulting to loss of convexity by projecting associated mappings in the state space.

The practical motivation for this study comes from the emergence of the above setting in several cases, such as, (i) approximations of nonlinear terms in linearisation of nonlinear systems, (ii) state dependent parametric uncertainties/disturbances and (iii) signals controlled by agents whose actions depend on the states, e.g., in game theoretic settings found in cybersecurity [9]. We focus on the computation of forward reachability set sequences, with the aim to characterise and compute the minimal robust positively invariant set, whose importance is significant for the performance and safety analysis of dynamical systems.

Our first main contribution is the establishment of *equivalence of systems subject to state dependent disturbances defined via polytopes in the extended state-disturbance space*

N.A. is with the School of Electronics, Electrical Engineering and Computer Science, Queen’s University Belfast, Northern Ireland, UK [n.athanasopoulos@qub.ac.uk](mailto:n.athanasopoulos@qub.ac.uk). E.V. is with the Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden [vlahakis@kth.se](mailto:vlahakis@kth.se). S.O. and C.T. are with Universite Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France, [sorin.olaru@centralesupelec.fr](mailto:sorin.olaru@centralesupelec.fr), [christopher.townsend@centralesupelec.fr](mailto:christopher.townsend@centralesupelec.fr).

*to piecewise polytopic affine systems.* The latter dynamics is no longer defined in a higher dimensional space, with the caveat that the description changes within pieces of a polytopic partition of the state space. We further show that this equivalent dynamics is convex in the support of the original disturbance set in the disturbances-states space. Thus, while for general piecewise polytopic affine dynamics there are negative complexity results [10] that extend to many classes of hybrid systems [11], in our setting convexity in the support of the disturbance set is appealing. Our second contribution is the establishment of a procedure to generate the aforementioned equivalent dynamics. We start from results on parametric polyhedral sets [12] revealing connections between parametric vertex and halfspace representations. We exploit properties of the coatomic lattice induced by polytopic projections of faces of the disturbance polytope, and define a partition of the state space, where in each piece the dynamics is polytopic affine. Our third contribution is the characterisation of the fixed points of sequences induced by the forward reachability maps, leading to the minimal invariant sets, by adopting our work [13] to the studied setting. Our proposed set maps are defined only in the state space. We observe in our setting the convexification arguments, e.g., [14] for approximating the minimal invariant set with a convex equivalent do not hold, however we show that the computation of a robust positively invariant set is possible under mild assumptions.

Sections II and III provide preliminaries and describe the transition from the halfspace representation to the vertex representation of parametric parameterised polytopes. Section IV formally establishes the equivalence between the original and the induced dynamics, Section V discusses the forward reachability set sequences of the equivalent dynamics, while conclusions are drawn in Section VI.

## II. PRELIMINARIES

Vector inequalities hold component wise. The convex hull of vectors  $v_1, \dots, v_q$  is  $\text{conv}(v_1, \dots, v_q)$ . A *face*  $\mathcal{F} \subseteq \mathcal{S}$  of a polytope  $\mathcal{S}$  is a set generated by the intersection of  $\mathcal{S}$  with any halfspace such that no interior point of  $\mathcal{S}$  lies in  $\mathcal{F}$ .  $\mathcal{S}$  and the empty set are also faces of the polytope. We call  $(n - 1)$ -dimensional faces of  $\mathcal{S}$  *facets*, and 0-dimensional faces *vertices*. The  $i$ th  $k$ -dimensional face of  $\mathcal{S}$  (or  $i$ th  $k$ -face of  $\mathcal{S}$ ) is  $\mathcal{F}_i^k(\mathcal{S})$ . For a set  $\mathcal{F}$ , its dimension  $\dim(\mathcal{F})$  is the dimension of the smallest subspace containing  $\mathcal{F}$ . The boundary, interior and closure of a set  $\mathcal{S}$  are  $\partial\mathcal{S}$ ,  $\text{int}(\mathcal{S})$  and  $\text{cl}(\mathcal{S})$ , respectively. A partially ordered set (poset)  $\mathcal{S}$  is a lattice if it is bounded, and every two elements  $x, y \in \mathcal{S}$

have a unique minimal upper bound (*join*  $x \vee y$ ) and a unique maximal lower bound (*meet*  $x \wedge y$ ). A lattice is coatomic if any element apart from the unique maximal element can be described as a meet of a finite number of elements, namely, coatoms. The projection of a set  $\mathcal{S} \subset \mathbb{R}^{n+m}$  on the subspace defined by the first  $n$  dimensions is  $\pi_{\mathbb{R}^n}(\mathcal{S}) = \{x \in \mathbb{R}^n : (\exists w \in \mathbb{R}^m : [x^\top \ w^\top]^\top \in \mathcal{S})\}$ . The product of a matrix  $M \in \mathbb{R}^{n \times m}$  with a set  $\mathcal{S} \subset \mathbb{R}^n$  is  $M\mathcal{S} = \{Mx : x \in \mathcal{S}\} \subseteq \mathbb{R}^n$ , and can be considered as a weighted projection when  $n < m$ . We denote the simplex in  $q$  dimensions as  $\mathcal{T}_q = \{\lambda \in \mathbb{R}^q : \lambda \geq 0, \lambda^\top \mathbf{1} = 1\}$ .

We consider disturbance sets  $\mathcal{W} \subset \mathbb{R}^{n+m}$  defined in the  $x-w$  space, defined together with selections in the  $w$ - and  $x$ - space, useful for the development of the results

$$\mathcal{W} = \left\{ \begin{bmatrix} x^\top & w^\top \end{bmatrix}^\top : G_x x + G_w w \leq d \right\}, \quad (1)$$

$$\mathcal{W}(x) = \left\{ w \in \mathbb{R}^m : \begin{bmatrix} x^\top & w^\top \end{bmatrix}^\top \in \mathcal{W} \right\}, \quad (2)$$

$$\text{supp}(\mathcal{W}) = \left\{ x \in \mathbb{R}^n : \left( \exists w \in \mathbb{R}^m : \begin{bmatrix} x^\top & w^\top \end{bmatrix}^\top \in \mathcal{W} \right) \right\}. \quad (3)$$

We note  $\text{supp}(\mathcal{W}) = \pi_{\mathbb{R}^n}(\mathcal{W})$ . The set  $\mathcal{W}(x)$  is a parameterised polyhedron in  $\mathbb{R}^m$ . We define the extension of  $\mathcal{W}$

$$\overline{\mathcal{W}} = \mathcal{W} \cup \left\{ \begin{bmatrix} x^\top & 0^\top \end{bmatrix}^\top : x \notin \text{supp}(\mathcal{W}) \right\}.$$

We consider discrete time linear inclusions, with exogenous signals whose range is bounded by a polytopical set

$$x^+ \in f_x(x, w), \quad w \in \overline{\mathcal{W}}(x), \quad (\Sigma_x)$$

with

$$f_x(x, w) = Ax + Bw,$$

where  $x \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^m$ ,  $A, B$  have appropriate dimensions,  $A$  is strictly stable, and  $\overline{\mathcal{W}}(x)$  is defined similarly as in (2). We denote a solution of  $(\Sigma_x)$  at time  $t$ , for an initial condition  $y \in \mathbb{R}^n$  and a feasible sequence  $\{w(i)\}$  as  $\phi(t; y, \{w(i)\})$ .

### III. EXPLICIT VERTEX REPRESENTATION

Work [12] establishes duality between the halfspace and vertex parametric representations. The set  $\mathcal{W}$  (1) is described by the convex hull of parameterised vertices  $\mathcal{W} = \text{conv}(v_1(x), v_2(x), \dots, v_q(x))$ , with each vertex being an affine function of the state  $x$

$$v_i(x) = C_i x + h_i, \quad i = 1, \dots, q. \quad (4)$$

We underline that each vertex  $v_i(x)$  is generally valid only in a subset of  $\text{supp}(\mathcal{W})$ . This is contrary to the non-parameterised case where the vertex representation is uniform. In the following, we summarize the procedure [12] for how to identify the vertices of the parameterised polyhedron  $\mathcal{W}(x)$  (2) and develop a framework to construct the polytopical partition  $\{\mathcal{C}_j\}$ ,  $j = 1, \dots, M$ , in which the parametric vertex representation is explicitly defined.

Let  $\mathcal{J}(x) := \{w \in \mathbb{R}^{n+m} : [I_n \ 0_m] w = x\}$  be an affine space of  $\mathbb{R}^{n+m}$ . The intersection of  $\mathcal{W}$  with  $\mathcal{J}(x)$  fixes the first  $n$  variables of  $\mathcal{W}$  to a constant state vector  $x$ , and if projected onto the  $w$  space, produces the *slice*

$\mathcal{W}(x)$ . We can thus alternatively write (2) as  $\mathcal{W}(x) = \pi_{\mathbb{R}^m}(\mathcal{W} \cap \mathcal{J}(x))$ , where the projection is performed here in the last  $m$  coordinates. We recall the result defining a parameterised vertex of  $\mathcal{W}(x)$ .

**Theorem 1** ([12]): Consider (1) and let  $x \in \text{supp}(\mathcal{W})$ . For each parameterised vertex (4)  $v_i(x)$  of  $\mathcal{W}(x) \subset \mathbb{R}^m$ , there exists an  $n$ -face  $\mathcal{F}_i^n(\mathcal{W})$  such that

$$v_i(x) = \pi_{\mathbb{R}^m}(\mathcal{F}_i^n(\mathcal{W}) \cap \mathcal{J}(x)).$$

By Theorem 1, a vertex  $v_i(x)$  appears if  $\mathcal{F}_i^n(\mathcal{W}) \cap \mathcal{J}(x) \neq \emptyset$ . As the state  $x$  varies in  $\text{supp}(\mathcal{W})$ , vertices may split, shift, and merge. The following Corollary gives the range of the state space for which a particular vertex exists.

**Corollary 1** ([12]): The range of the state  $x$  over which  $v_i(x)$  exists is  $\pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W}))$ .

We note that not all the  $n$ -faces correspond to a parameterised vertex. This is stated next.

**Theorem 2** ([12]): For each  $n$ -face  $\mathcal{F}_i^n(\mathcal{W})$ , and for the set of all points  $\begin{bmatrix} x^\top & w^\top \end{bmatrix}^\top \in \mathcal{F}_i^n(\mathcal{W})$ , one of the following is true:

- (i)  $w$  is an affine function of  $x$ , i.e.,  $w = v_i(x)$ , where  $v_i(x)$  is a parameterised vertex of  $\mathcal{W}(x)$ ,
- (ii)  $w$  is not constrained, i.e., for a given value of  $x$ , more than one value of  $w$  is feasible.

Roughly, condition (ii) of Theorem 2 relates to the case where the projection of the  $n$ -face  $\mathcal{F}_i^n(\mathcal{W})$  in the  $x$ -space  $\pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W}))$  lies in a strict subspace of  $\mathbb{R}^n$ . To show how to identify the sets  $\mathcal{C}_j$ ,  $j = 1, \dots, M$ , where in each set the vertex representation of  $\mathcal{W}(x)$  is unique, we construct a poset. To this purpose, consider sets  $P_i \subset \mathbb{R}^n$ ,  $\dim(P_i) = n$ ,  $i = 1, \dots, N$ . Consider the poset  $L(P)$ , ordered by the set inclusion  $\subseteq$ , with maximal element  $P = \bigcup_{i=1}^N P_i$  and minimal element  $\emptyset$ . Apart from the maximal and minimal element of  $L(P)$ , each element  $L \in L(P)$  is constructed as the intersection of some finite number  $Q$  of sets  $P_i$ ,  $i = 1, \dots, N$ ,  $Q \leq N$ , so that  $L = \bigcap_{i=1}^Q P_i$  and  $\dim(L) = n$ . We define the rank of each element  $L$  as the number of pieces  $P_i$  whose intersection defines the element, i.e.,  $\text{rank}(L) = \left\{ M \in \mathbb{N} : L = \bigcap_{i=1}^M P_i \right\}$ . Last, we call an element  $\hat{L}$  of  $L(P)$  to be a *leaf* of  $L(P)$  if it is not an upper bound of any other element  $L \in L(P)$ , excluding the empty set.

**Proposition 1:** Consider  $\mathcal{W}$  (1), and all  $n$ -faces  $\mathcal{F}_i^n(\mathcal{W})$ ,  $i = 1, \dots, N$ ,  $N > 0$ , satisfying the condition of Theorem 2(i). Consider the above construction of the poset  $L(P)$ , setting

$$P_i = \pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W})), \quad i = 1, \dots, N. \quad (5)$$

Moreover, let  $\hat{L} = \{\hat{L}_{q_1}, \dots, \hat{L}_{q_M}\}$  be the set of leaves of  $L(P)$ . The following hold:

- (i)  $L(P)$  is a coatomic lattice.
- (ii) For any two leaves  $\hat{L}_1, \hat{L}_2$ , it holds  $\dim(\hat{L}_1 \cap \hat{L}_2) < n$ .
- (iii) It holds that  $P = \bigcup_{j=1}^M \hat{L}_{q_j}$ .

**Proof** (i) For each element  $L \in L(P)$  there exists the trivial lower bound  $\emptyset \subset L$  and upper bound  $L \subset P$ . Suppose for two sets  $L_i, L_j$ , there is a lower bound  $L_k \neq \emptyset$ , so that  $L_k \subseteq L_i \cap L_j$ . Suppose  $k_i = \text{rank}(L_i)$ ,  $k_j = \text{rank}(L_j)$ , with  $k_i \leq k_j$ . By assumption,  $L_i \cap L_j \neq \emptyset$ ,  $\dim(L_i \cap L_j) = n$ .

Also, by construction of the poset  $L(P)$ , there is  $Q \geq 1$  so that  $L_i \cap L_j = L_i \cap (\cap_{l=1}^Q P_{il})$ , with  $1 \leq i_l \leq N$ ,  $l = 1, \dots, Q$ . However, then necessarily  $L_k = L_i \cap L_j$  is an element of  $L(P)$ , which makes it a unique maximal lower bound for  $L_i, L_j$ . To show there is a unique minimal upper bound, suppose for two sets  $L_i, L_j$  there are two upper bounds  $L_{k_1}, L_{k_2}$ , so that  $L_{k_1} \supseteq L_i \cup L_j$ ,  $L_{k_2} \supseteq L_i \cup L_j$ , and that  $L_{k_1} \not\supseteq L_{k_2}$ ,  $L_{k_2} \not\supseteq L_{k_1}$ . However, necessarily the set  $L_k = L_{k_1} \cap L_{k_2} \in L(P)$  as it is the intersection of a finite number of pieces  $P_i$ , and moreover  $L_k \supset L_{k_1}$ ,  $L_k \supset L_{k_2}$ , thus, a unique minimal upper bound exists. Thus,  $L(P)$  is a lattice. To show  $L(P)$  is coatomic, we observe that the elements  $P_i$ ,  $i = 1, \dots, N$  are the coatoms.

(ii) Suppose there are two leaves  $\hat{L}_1, \hat{L}_2$ , with  $k_1 = \text{rank}(\hat{L}_1)$ ,  $k_2 = \text{rank}(\hat{L}_2)$ ,  $k_1 \leq k_2$ , such that  $\dim(\hat{L}_1 \cap \hat{L}_2) = n$ . As in the proof of statement (i), necessarily there is a subset of coatoms  $Q \leq k_2 - k_1$  so that  $\hat{L}_1 \cap \hat{L}_2 = \hat{L}_1 \cap (\cap_{i=1}^Q P_{i1})$ . Then, it holds that the set  $L^* = \hat{L}_1 \cap P_{i1} \in L(P)$ , and since  $L^* \subset \hat{L}_1$ ,  $\hat{L}_1$  is not a leaf and a contradiction has been reached.

(iii) Suppose that  $\cup_{i=1}^{\hat{M}} \hat{L}_{qj} \subset P$ , and there is an element  $L^* \in L(P)$  so that  $L^* \cup (\cup_{i=1}^{\hat{M}} \hat{L}_{qj}) = P$ . Since  $L(P)$  is coatomic, there are  $Q$  coatoms so that  $L^* = \cap_{i=1}^Q P_{qi}$ . Since  $L^*$  is not a leaf, there is at least a piece  $P_{qQ+1}$  so that  $\dim(L^* \cap P_{qQ+1}) = n$ . Let  $\bar{L} = L^* \setminus P_{qQ+1}$ ,  $\dim(\bar{L}) = n$ . Take  $x \in \partial L^* \cap \partial P_{qQ+1}$ , and choose  $w_x \in \mathbb{R}^m$  so that  $[x^\top \ w_x^\top]^\top \in (\cap_{i=1}^Q \partial \mathcal{F}_{qi}^n(\mathcal{W})) \cap \mathcal{F}_{qQ+1}^n(\mathcal{W})$  and is also a vertex of  $\mathcal{W}$ . Also, take  $y \in \partial L^* \setminus P_{qQ+1}$ , and  $w_y$  so that  $[y^\top \ w_y^\top]^\top \in \cap_{i=1}^{Q^*} \mathcal{F}_{qi}^n(\mathcal{W})$ , for some  $Q^* \leq Q$ , and it is a vertex of  $\mathcal{W}$ . Consider  $\mathcal{F}_\bullet^1(\mathcal{W})$  to denote the 1-face of  $\mathcal{W}$  that is formed by the vertices  $[x^\top \ w_x^\top]^\top$ ,  $[y^\top \ w_y^\top]^\top$  and let  $\mathcal{F}_\bullet^n(\mathcal{W})$  be an  $n$ -face,  $n \geq 1$  that contains  $\mathcal{F}_\bullet^1(\mathcal{W})$ . Take  $z = \frac{x+y}{2}$ , so that  $z \in \pi(\mathcal{F}_\bullet^n(\mathcal{W})) = P_\bullet$ . However, since  $z \in \text{int}(\bar{L})$ , a contradiction has been reached as it implies  $P_\bullet \cap \bar{L} \neq \emptyset$ . ■

**Theorem 3:** Consider  $\mathcal{W} \subset \mathbb{R}^{n+m}$  (1), the lattice  $L$  as defined in Proposition 1, and its set of leaves  $\hat{L}$ . It holds

$$\text{supp}(\mathcal{W}) = \bigcup_{i=1}^{\hat{M}} \hat{L}_{qi}. \quad (6)$$

**Proof** From Proposition 1(iii), we have that  $\bigcup_{j=1}^{\hat{M}} \hat{L}_{qj} = \bigcup_{i=1}^N P_i$ , with  $P_i = \pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W}))$ ,  $i = 1, \dots, N$  defined in (5). What remains to show is that the support of  $\mathcal{W}$  is covered completely by the union of the projections of  $n$ -faces of  $\mathcal{W}$ , i.e.,  $\text{supp}(\mathcal{W}) = \bigcup_{i=1}^N P_i$ . Let  $m = 1$ , and let any fixed  $n \geq 1$ . Consequently,  $\mathcal{F}_i^n(\mathcal{W})$  are the facets of  $\mathcal{W}$  with the property that  $\bigcup_{i=1}^N \mathcal{F}_i^n(\mathcal{W}) = \partial \mathcal{W}$ . For any  $x \in \text{supp}(\mathcal{W})$ , choose a vector  $w^* \in \mathbb{R}$  such that  $[x^\top \ (w^*)^\top]^\top \in \partial \mathcal{W}$ . Necessarily, there is an  $1 \leq i^* \leq N$  such that  $[x^\top \ (w^*)^\top]^\top \in \mathcal{F}_{i^*}^n(\mathcal{W})$ , thus,  $x \in \pi_{\mathbb{R}^n}(\mathcal{F}_{i^*}^n(\mathcal{W}))$ . Suppose the statement of the theorem holds for  $m = k$ , and any fixed  $n \geq 1$ . Let  $m = k + 1$ , and any fixed  $n \geq 1$ . Set  $\bar{m} = k$ ,  $\bar{n} = n + 1$  so that  $\mathbb{R}^{\bar{n}}$  corresponds to  $\bar{x} = [x^\top \ w_1]^\top$ , and  $\mathbb{R}^{\bar{m}}$  corresponds to  $\bar{w} = [w_2 \dots w_{k+1}]^\top$ . Take any  $\bar{x}$  in the set  $\{\bar{x} \in \mathbb{R}^{\bar{n}} : [\bar{x}^\top \ \bar{w}^\top]^\top \in \mathcal{W}\}$ . Then,

by assumption, there is an  $\bar{n}$ -face  $\mathcal{F}_{i^*}^{\bar{n}}(\mathcal{W})$  of  $\mathcal{W}$  so that  $\bar{x} \in \mathcal{S}$ , where  $\mathcal{S} = \pi_{\mathbb{R}^{\bar{n}}}(\mathcal{F}_{i^*}^{\bar{n}}(\mathcal{W}))$ . The set  $\mathcal{S}$  is a polytope since it is the projection of a face of a polytopical set, see, e.g., [15]. Moreover, by compactness of  $\mathcal{W}$  we can choose  $\mathcal{F}_{i^*}^{\bar{n}}(\mathcal{S})$  such that  $\dim \mathcal{S} = n + 1$ . Choose  $w_1^*$  such that  $\bar{x}^* = [x^\top \ w_1^*]^\top \in \partial \mathcal{S}$ . Necessarily, such  $w_1^*$  exists by compactness of  $\mathcal{S}$ . Thus, there exists an  $n$ -face, i.e., a facet, of  $\mathcal{S}$ , namely,  $\mathcal{F}_{i^*}^n(\mathcal{S})$  so that  $\bar{x}^* \in \mathcal{F}_{i^*}^n(\mathcal{S})$ . Since both  $\mathcal{S}$ ,  $\mathcal{F}_{i^*}^n(\mathcal{S})$  are faces of  $\mathcal{S}$  and  $\mathcal{F}_{i^*}^n(\mathcal{S}) \subset \mathcal{S}$ , it holds that there is a face of  $\mathcal{F}_{i^*}^{\bar{n}}(\mathcal{W})$ , say,  $\mathcal{F}_{i^*}^{n^*}(\mathcal{W}) \subset \mathcal{F}_{i^*}^{\bar{n}}(\mathcal{W})$ , with  $n^* \leq \bar{n} - 1$ , or,  $n^* \leq n$ , such that  $\mathcal{F}_{i^*}^{n^*}(\mathcal{S}) = \pi_{\mathbb{R}^n}(\mathcal{F}_{i^*}^{n^*}(\mathcal{W}))$  [15, Lemma 7.10]. Consequently, and summarising, there is an  $n$ -face  $\mathcal{F}_i^n(\mathcal{W})$  for which  $x \in \pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W}))$ . Thus, since  $x$  is chosen arbitrarily,  $\text{supp}(\mathcal{W}) = \bigcup_{i=1}^N P_i = \bigcup_{j=1}^{\hat{M}} \hat{L}_{qj}$ , thus statement (6) holds. ■

Proposition 1 and Theorem 3 induce a construction of a complete polyhedral partition  $\{\mathcal{C}_j\}$ ,  $j = 1, \dots, \hat{M}$ , of  $\text{supp}(\mathcal{W})$ , by considering the  $n$ -faces of  $\mathcal{W}$ ,  $\mathcal{F}_i^n(\mathcal{W})$ ,  $i = 1, \dots, N$ , that satisfy the condition Theorem 2(i). To partition the entire  $x$  space, we need to partition the area  $\mathcal{W}' = \mathbb{R}^n \setminus \text{supp}(\mathcal{W})$  as well. Let  $\mathcal{C}_j$ ,  $j = \hat{M} + 1, \dots, M$ , with  $\bigcup_{j=\hat{M}+1}^M \mathcal{C}_j \equiv \mathcal{W}'$ , represent  $M - \hat{M}$  non-overlapping, convex partitions of  $\mathcal{W}'$ . Then,  $\mathcal{C}_j$ ,  $j = 1, \dots, M$ , are  $M$  non-overlapping, convex partitions of the  $x$  space  $\mathbb{R}^n$ . We summarize the partitioning procedure in Algorithm 1.

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**Algorithm 1** Partitioning of  $x$  space  $\mathbb{R}^n$ .

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- 1: Compute  $\mathcal{W}$  from  $\mathcal{W}(x)$ , and identify all  $\mathcal{F}_i^n(\mathcal{W})$ .
  - 2: Let  $\mathcal{F}_i^n(\mathcal{W})$ ,  $i = 1, \dots, N$ , satisfy Theorem 2(i).
  - 3: Construct poset  $L$  with maximal element  $P = \bigcup_{i=1}^N P_i$ , where  $P_i = \pi_{\mathbb{R}^n}(\mathcal{F}_i^n(\mathcal{W}))$ , as in Proposition 1.
  - 4: Identify the set of leaves  $\hat{L} = \{\mathcal{C}_1, \dots, \mathcal{C}_{\hat{M}}\}$ .
  - 5: Define  $\mathcal{W}' = \mathbb{R}^n \setminus \text{supp}(\mathcal{W})$ .
  - 6: Construct  $M - \hat{M}$  non-overlapping, convex partitions of  $\mathcal{W}'$ , namely,  $\mathcal{C}_j$ ,  $j = \hat{M} + 1, \dots, M$ .
  - 7: Return  $\mathcal{C}_1, \dots, \mathcal{C}_{\hat{M}}, \mathcal{C}_{\hat{M}+1}, \dots, \mathcal{C}_M$ .
- 

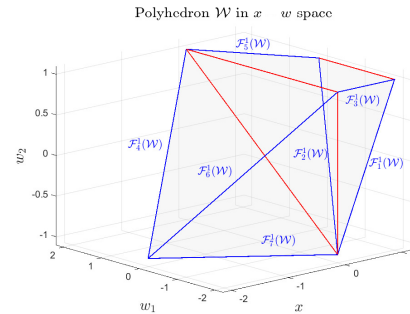


Fig. 1. Set  $\mathcal{W}$  (1) for Example 1, with  $\mathcal{F}_i^1(\mathcal{W})$  in blue and red satisfying Theorem 2(i) and Theorem 2(ii), respectively.

**Example 1:** Let  $\mathcal{W}(x) = \{w \in \mathbb{R}^2 : [x \ w^\top]^\top \in \mathcal{W}\}$  be a parameterised polyhedron, with its non-parameterised description,  $\mathcal{W} \subset \mathbb{R}^3$ , shown in Fig. 1. Slices of  $\mathcal{W}$ ,  $\mathcal{W}(x)$ , for two values of  $x \in \mathbb{R}$  are shown in Fig. 4. We partition

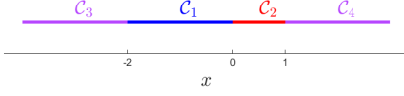


Fig. 2. Partitions of  $\mathbb{R}$  for Example 1, with blue and red referring to partition of  $\text{supp}(\mathcal{W})$  and with magenta referring to  $\mathcal{W}'$ .

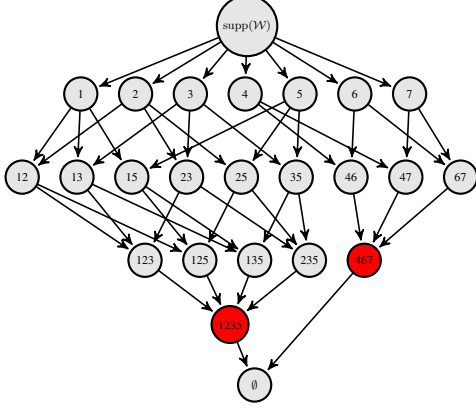


Fig. 3. Poset  $L$  for Example 1, where the leaves are in red, with the indices 467, 1235, denoting the leaves  $L = P_4 \cap P_6 \cap P_7$ ,  $L = P_1 \cap P_2 \cap P_3 \cap P_5$ , respectively.

the  $x$ -space,  $\mathbb{R}$ , based on Algorithm 1. The polytope  $\mathcal{W}$  has eleven 1-faces from which seven, denoted as  $\mathcal{F}_i^1(\mathcal{W})$ ,  $i = 1, \dots, 7$ , satisfy the condition Theorem 2(i) and constitute the coats in the lattice of Fig. 3. Let  $P = \bigcup_{i=1}^7 P_i$ , with  $P_i = \pi_{\mathbb{R}}(\mathcal{F}_i^1(\mathcal{W}))$ ,  $i = 1, \dots, 7$ . We construct the lattice  $L$  in Fig. 3 as in Proposition 1, and identify its set of leaves  $\hat{L} = \{P_4 \cap P_6 \cap P_7, P_1 \cap P_2 \cap P_3 \cap P_5\}$ , which leads to two partitions in the support of  $\mathcal{W}$ , namely,  $\mathcal{C}_1 = [-2, 0]$  and  $\mathcal{C}_2 = [0, 1]$ . Since  $\mathcal{W}' = \mathbb{R} \setminus \text{supp}(\mathcal{W})$ , with  $\text{supp}(\mathcal{W}) = [-2, 1]$ , we partition  $\mathcal{W}'$  by  $\mathcal{C}_3 = (-\infty, -2]$  and  $\mathcal{C}_4 = [1, \infty)$ , and the entire  $x$  space  $\mathbb{R} = \bigcup_{j=1}^4 \mathcal{C}_j$ . The partition is shown in Fig. 2.

#### IV. EQUIVALENCE OF SYSTEMS

Consider the set  $\mathcal{W}$ , the induced partition  $\{\mathcal{C}_i\}$  of  $\mathbb{R}^n$  consisting of  $M$  pieces, as created in Algorithm 1. For each piece  $\mathcal{C}_i$ , we rename the set of active vertices (4) to  $\{v_i^j(x)\}$ , with  $v_i^j(x) = C_i^j x + h_i^j$ ,  $j = 1, \dots, q_i$ . We consider the system defined by the difference inclusion

$$z^+ \in f_z(z, \lambda), \quad z \in \mathcal{C}_i, \quad \lambda \in \mathcal{T}_{q_i}, \quad (\Sigma_z)$$

where

$$f_z(z, \lambda) = A_i(\lambda)z + b_i(\lambda),$$

$$A_i(\lambda) = \sum_{j=1}^{q_i} A_i^j \lambda_j, \quad b_i(\lambda) = \sum_{j=1}^{q_i} b_i^j \lambda_j, \quad (7)$$

$$A_i^j = A + BC_i^j, \quad b_i^j = Bh_i^j, \quad i = 1, \dots, M, \quad j = 1, \dots, q_i. \quad (8)$$

We note for the pieces  $\mathcal{C}_i$  not in the support of  $\mathcal{W}$ , i.e., when  $\text{int}(\mathcal{C}_i) \cap \text{int}(\text{supp}(\mathcal{W})) = \emptyset$ , we have  $q_i = 1$ ,  $C_i^1 = 0$ ,  $h_i^1 = 0$ . We state the main result of the note.

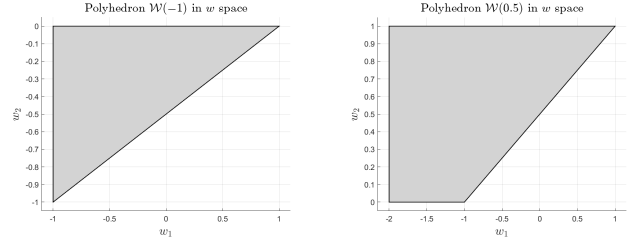


Fig. 4. Slices  $\mathcal{W}(x)$  in the  $w$  space of  $\mathcal{W}$  for  $x = -1$  and  $x = 0.5$  for Example 1.

**Theorem 4:** Consider the systems described by the inclusions  $(\Sigma_x)$ ,  $(\Sigma_z)$ . Let  $y \in \mathbb{R}^n$  and set

$$x(0) = z(0) = y.$$

Let  $x(t) = \phi(t; y, \{w(i)\})$ ,  $t \geq 0$ , denote a solution of  $(\Sigma_x)$  for a choice of a valid sequence  $w(i) \in \mathcal{W}(x(i))$ ,  $i = 1, \dots, t$ . Then, there exists a sequence  $\{\lambda(i)\}$ ,  $\lambda(i) \in \mathcal{T}_{q_{j(i)}}$ ,  $z(i) \in \mathcal{C}_{j(i)}$ , such that  $z(t) = \phi_z(t; y, \{\lambda(i)\})$ , with  $z(i) \in \mathcal{C}_{j(i)}$ , and

$$x(t) = z(t), \quad \text{for all } t \geq 0.$$

**Proof** Both inclusions  $(\Sigma_x)$ ,  $(\Sigma_z)$  are time invariant, thus it is sufficient to show that there exists a vector  $\lambda$  such that  $f_z(y, \lambda) = f_x(y, w)$  for any choice of  $w^* \in \overline{\mathcal{W}}$ . Indeed, for any  $y \in \mathbb{R}^n$ , and since the partition is complete by Theorem 3, i.e.,  $\bigcup_i \mathcal{C}_i = \mathbb{R}^n$ , there is  $i^*$  so that  $y \in \mathcal{C}_{i^*}$ . We consider two cases: If  $\mathcal{C}_{i^*} \notin \text{supp}(\mathcal{W})$ , then trivially  $w^* = 0$ , and  $\lambda = 1$ . If  $\mathcal{C}_{i^*} \in \text{supp}(\mathcal{W})$ , then for any  $w^* \in \mathcal{W}(y)$  there is at least one vector  $\lambda^* \in \mathcal{T}_{q_{i^*}}$  satisfying  $w = \sum_{j=0}^{i^*} \lambda_j^* v_j(y)$ , with  $v_j(y) = C_{i^*}^j y + h_{i^*}^j$ ,  $j = 1, \dots, q_{i^*}$ . Consequently,  $f_x(y, w^*) = Ay + B \sum_{i=0}^{i^*} \lambda_i^* v_i(y) = \sum_{j=0}^{i^*} A_i^j \lambda_j^* y + \sum_{j=0}^{i^*} b_i^j \lambda_j^* y = f_z(y, \lambda^*)$ . ■

It is straightforward to show that the converse statement of Theorem 4 also holds, namely, for any choice of  $y \in \mathcal{C}_i$  and  $\lambda^* \in \mathcal{T}_{q_i}$ , there is (in this case exactly one)  $w^* \in \mathcal{W}(y)$  so that  $f_z(y, \lambda) = f_x(y, w)$ .

#### V. FORWARD REACHABILITY MAPS AND MINIMAL INVARIANT SET COMPUTATION

Additional to providing insight, Theorem 4 can have a computational significance. In this section we investigate how to compute forward reachability set sequences, using the system dynamics  $(\Sigma_z)$ . For a set  $\mathcal{S} \subset \mathbb{R}^n$ , we consider the maps

$$\mathbb{F}(\mathcal{S}) = \{f_x(x, w); x \in \mathcal{S}, w \in \overline{\mathcal{W}}(x)\}, \quad (9)$$

$$\mathbb{F}_x(\mathcal{S}) = [A \quad B] ((\mathcal{S} \times \mathbb{R}^m) \cap \overline{\mathcal{W}}), \quad (10)$$

$$\mathbb{F}_z(\mathcal{S}) = \{f_z(x, \lambda); x \in \mathcal{S}, \lambda \in \mathcal{T}\}. \quad (11)$$

**Lemma 1:** For any set  $\mathcal{S} \subset \mathbb{R}^n$ , it holds

$$\mathbb{F}(\mathcal{S}) = \mathbb{F}_x(\mathcal{S}) = \mathbb{F}_z(\mathcal{S}). \quad (12)$$

**Proof**  $\mathbb{F}(\mathcal{S}) = \mathbb{F}_z(\mathcal{S})$  follows directly from Theorem 4. Taking into account that  $\mathbb{F}(\mathcal{S}) = \mathbb{F}_x(\mathcal{S})$  from [13, Section II.B], (12) follows. ■

$\mathbb{F}(\mathcal{S})$  cannot be computed analytically. On the one hand, the affine map  $\mathbb{F}_x(\mathcal{S})$  [13] is a projection to the  $x$ -space, which

can be costly even when  $\mathcal{S}$  is a polytope, and, e.g., Fourier-Motzkin elimination or variations are used [16]. On the other hand,  $\mathbb{F}_z(\mathcal{S})$  requires maps in the state space only, and in combination with the following result it suggests that  $\mathbb{F}_z(\mathcal{S})$  could offer a computationally appealing alternative.

*Lemma 2:* Consider a convex set  $\mathcal{S} \subseteq \text{supp}(\mathcal{W})$ . It holds

$$\mathbb{F}_z(\mathcal{S}) = \text{conv}(\mathbb{F}_z(\mathcal{S})). \quad (13)$$

**Proof** Since  $\mathcal{S} \subseteq \text{supp}(\mathcal{W})$ , it holds that  $(\mathcal{S} \times \mathbb{R}^m) \cap \overline{\mathcal{W}} = (\mathcal{S} \times \mathbb{R}^m) \cap \mathcal{W}$ , and since the projection of a polytopic set remains a polytope, e.g., [15], the map  $\mathbb{F}_x(\mathcal{S})$  (10) is a polytopic set in  $\mathbb{R}^n$ . Consequently, by Lemma 1, it holds that  $\mathbb{F}_z(\mathcal{S}) = \mathbb{F}_x(\mathcal{S}) = \text{conv}(\mathbb{F}_x(\mathcal{S})) = \text{conv}(\mathbb{F}_z(\mathcal{S}))$ . ■

The following Corollary suggests it is possible to simplify the reachable set computation by clustering it to two groups.

*Corollary 2:* Consider the convex set  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ , where  $\mathcal{S}_1 = \mathcal{S} \cap \text{supp}(\mathcal{W})$ ,  $\mathcal{S}_2 = \text{cl}(\mathcal{S} \setminus \mathcal{S}_1)$ . It holds

$$\mathbb{F}_z(\mathcal{S}) = \mathbb{F}_z(\mathcal{S}_2) \cup \text{conv}(\mathbb{F}_z(\mathcal{S}_1)). \quad (14)$$

We call a set  $\mathcal{S}$  robust positively invariant (RPI) with respect to the system  $(\Sigma_x)$  if  $x \in \mathcal{S}$  implies  $Ax + Bw \in \mathcal{S}$ , for any  $w \in \overline{\mathcal{W}}(x)$ . We call  $\mathcal{S}_m$  the minimal RPI (mRPI) if it is RPI and contained in any other RPI. The companion paper [13] examines the general case of a set  $\mathcal{W}$ , and highlights, contrary to the case of state-independent disturbances, that the existence of a minimal robust positively invariant set (mRPI) is not guaranteed. In the following result, we show how under a relevant assumption existence of mRPI set is guaranteed.

*Proposition 2:* Consider the set  $\mathcal{W}$  (1), and let  $\{0\} \in \mathcal{W}(x)$ , for all  $x \in \text{supp}(\mathcal{W})$ . Then, the mRPI  $\mathcal{S}_m$  exists, and is the limit of the sequence

$$\mathcal{R}_{i+1} = \mathbb{F}_z(\mathcal{R}_i), \quad \mathcal{R}_0 = \{0\}. \quad (15)$$

**Proof** From [13, Theorem 13], existence of the mRPI is guaranteed. To show it is the limit of the sequence (15), one can consider [13, Lemma 15], observing that for any  $\mathcal{R}_i$ ,  $i \geq 1$  it holds  $\mathcal{R}_i = \bigcup_{k=0}^i \mathcal{R}_k$ . ■

We can efficiently compute the elements of the set sequence (15) utilising the set map (11). For the initialisation of  $\mathcal{R}_0 = \{0\}$ , it is straightforward to see that the elements  $\mathcal{R}_i$ ,  $i \geq 0$ , can be expressed as unions of polytopes.

*Proposition 3:* Consider  $\mathcal{W}$ , the corresponding partition induced by Algorithm 1, and the system  $(\Sigma_z)$ . Consider the set  $\mathcal{S} = \bigcup_{k=1}^p \mathcal{S}_k$ , where each  $\mathcal{S}_k$ ,  $k = 1, \dots, p$  is a convex set, and let  $\mathcal{S}_k^w = \mathcal{S}_k \cap \text{supp}(\mathcal{W})$ ,  $\mathcal{S}_k^{\overline{w}} = \text{cl}(\mathcal{S}_k \setminus \text{supp}(\mathcal{W}))$ ,  $k = 1, \dots, p$ . Then, it holds

$$\mathbb{F}_z(\mathcal{S}) = \left[ \bigcup_{k=1}^p \text{conv} \left\{ \bigcup_{i=1}^{\hat{M}} \bigcup_{j=1}^{q_i} \left( A_i^j(\mathcal{S}_k^w \cap \mathcal{C}_i) \oplus b_i^j \right) \right\} \right] \cup \left[ \bigcup_{k=1}^p \bigcup_{i=\hat{M}+1}^M A(\mathcal{S}_k^{\overline{w}} \cap \mathcal{C}_i) \right]. \quad (16)$$

**Proof** We have  $\mathbb{F}_z(\mathcal{S}) = (\bigcup_{k=1}^p \mathbb{F}_z(\mathcal{S}_k^w)) \cup (\bigcup_{k=1}^p \mathbb{F}_z(\mathcal{S}_k^{\overline{w}}))$ . The second term corresponds to the second line of equation (16), since in each piece  $\mathcal{C}_i$ ,  $i = \hat{M} + 1, \dots, M$  it holds  $f_z(x, \lambda) = Ax$ . The first line of

equation (16) is derived by taking into account that each set  $\mathcal{S}_k^w$  is convex, and, by Lemma 2, for any  $k = 1, \dots, p$ , it holds

$$\begin{aligned} \text{conv}(\mathbb{F}_z(\mathcal{S}_k^w)) &= \text{conv} \left\{ \bigcup_{i=1}^{\hat{M}} \mathbb{F}_z(\mathcal{S}_k^w \cap \mathcal{C}_i) \right\} \\ &= \text{conv} \left\{ \bigcup_{i=1}^{\hat{M}} \text{conv} \left( \bigcup_{j=1}^{q_i} \left( A_i^j(\mathcal{S}_k^w \cap \mathcal{C}_i) \oplus b_i^j \right) \right) \right\} \\ &= \text{conv} \left\{ \bigcup_{i=1}^{\hat{M}} \bigcup_{j=1}^{q_i} \left( A_i^j(\mathcal{S}_k^w \cap \mathcal{C}_i) \oplus b_i^j \right) \right\}. \end{aligned}$$

The fact that for each piece  $\mathcal{C}_i \in \text{supp}(\mathcal{W})$  it holds that  $\mathbb{F}_z(\mathcal{S}_k^w \cap \mathcal{C}_i) = \text{conv}(\bigcup_{j=1}^{q_i} (A_i^j(\mathcal{S}_k^w \cap \mathcal{C}_i) \oplus b_i^j))$  follows from the definition of the dynamics  $(\Sigma_z)$ , consistent with the forward reachability computation for linear polytopic systems, see, e.g., [1]. ■

Proposition 3 and map (16) allow the elimination of the ‘for all’ quantifier for the variable  $\lambda$ , which in turn eliminated the ‘for all’ quantifier for  $w \in \mathcal{W}(x)$  by Lemma 1. Contrary to convexification procedures for polytopic linear systems [14], [1], it is not possible to obtain similar results for the systems under study, as the state dependent switching of the dynamics  $(\Sigma_z)$ , or equivalently, the state dependent disturbances of the original dynamics  $(\Sigma_x)$  do not in general induce convex maps. Nevertheless, for the setting of this paper, we can define a convexified set sequence of (15) converging to a robust positively invariant set, however not necessarily the minimal.

*Proposition 4:* Consider the set  $\mathcal{W}$  (1), and let  $\{0\} \in \mathcal{W}(x)$ , for all  $x \in \text{supp}(\mathcal{W})$ . Consider the set sequence  $\{\mathcal{R}_i^c\}$ ,

$$\mathcal{R}_{i+1}^c = \text{conv}(\mathbb{F}_z(\mathcal{R}_i^c)), \quad \mathcal{R}_0^c = \{0\}. \quad (17)$$

Then, the set sequence converges to a robust positively invariant (RPI)  $\mathcal{S}_m^c$ .

**Proof** By compactness of the set  $\mathcal{W}$ , there is a number  $K$  such that  $\mathcal{W}(x) \subseteq \mathbb{B}(K, 0)$ , for all  $x \in \text{supp}(\mathcal{W})$ , where  $\mathbb{B}(K, 0) \subset \mathbb{R}^n$  denotes the  $n$ -th dimensional ball centered at the origin with radius  $K$ . Consider the system

$$x^+ = Ax + Bw, \quad w \in \mathbb{B}(K, 0). \quad (18)$$

Since (18) is subject to state-independent disturbances, by utilising results, e.g., [17], the set sequence  $\mathcal{R}_{i+1}^b = A\mathcal{R}_i^b \oplus \mathbb{B}(K, 0)$ , with  $\mathcal{R}_0^b = \{0\}$  converges to the convex mRPI set  $\mathcal{S}_m^b$ . By construction,  $\mathcal{R}_i^b \supseteq \mathcal{R}_i^c$  for any  $i \geq 0$ , and taking into account that the set sequence (17) is nested, i.e.,  $\mathcal{R}_i^c \subseteq \mathcal{R}_{i+1}^c$ , for all  $i \geq 0$ , it follows that (17) converges to a fixed point  $\mathcal{S}_m^c$ . To show that  $\mathcal{S}_m^c$  is invariant, it is sufficient to observe that  $\mathcal{S}_m^c = \bigcup_{i=0}^{\infty} \mathcal{R}_i^c$ . ■

The following Corollary highlights the case when the convex mRPI can be retrieved from (17).

*Corollary 3:* Consider the set  $\mathcal{W}$  (1), and let  $\{0\} \in \mathcal{W}(x)$ , for all  $x \in \text{supp}(\mathcal{W})$ , and the set sequence (17). Suppose that  $\mathcal{S}_m \subseteq \text{supp}(\mathcal{W})$ . Then,  $\mathcal{S}_m^c = \text{conv}(\mathcal{S}_m)$ .

The following example illustrates the developed results of this section.

**Example 2:** Consider the system  $(\Sigma_x)$ , with  $A = \begin{bmatrix} 0 & 0.83 \\ -0.83 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The disturbance set  $\mathcal{W} \subset \mathbb{R}^3$  is a parameterized polyhedron, which is represented in the state-disturbance space in Fig. 5 left. To compute the mRPI set, we first utilise Algorithm 1 to recover the partition of  $\mathbb{R}^2$ . In Fig. 5, left, we highlight in red the 2-faces of  $\mathcal{W}$ , namely,  $\mathcal{F}_i^2(\mathcal{W})$ ,  $i = 1, 2, 3, 4$ , that satisfy Theorem 2i). We recall that the projections  $\pi_{\mathbb{R}^2}(\mathcal{F}_i^2(\mathcal{W}))$ ,  $i = 1, 2, 3, 4$ , onto  $\mathbb{R}^2$ , are the coatoms  $P_i$  of the lattice  $L(P)$  shown in Fig. 6. We retrieve the partitions of  $\text{supp}(\mathcal{W})$ , with  $\mathcal{C}_1 = \pi_{\mathbb{R}^2}(\mathcal{F}_1^2(\mathcal{W})) \cap \pi_{\mathbb{R}^2}(\mathcal{F}_2^2(\mathcal{W}))$ ,  $\mathcal{C}_2 = \pi_{\mathbb{R}^2}(\mathcal{F}_1^2(\mathcal{W})) \cap \pi_{\mathbb{R}^2}(\mathcal{F}_4^2(\mathcal{W}))$ ,  $\mathcal{C}_3 = \pi_{\mathbb{R}^2}(\mathcal{F}_2^2(\mathcal{W})) \cap \pi_{\mathbb{R}^2}(\mathcal{F}_3^2(\mathcal{W}))$ , and  $\mathcal{C}_4 = \pi_{\mathbb{R}^2}(\mathcal{F}_3^2(\mathcal{W})) \cap \pi_{\mathbb{R}^2}(\mathcal{F}_4^2(\mathcal{W}))$ , which are shown in Fig. 5, right, in gray. The partitions of  $\mathcal{W}' = \mathbb{R}^2 \setminus \text{supp}(\mathcal{W})$ , denoted as  $\mathcal{C}_j$ ,  $j = 5, 6, 7, 8$ , are shown in the same figure in light gray completing the partition of the entire  $\mathbb{R}^2$ . Having identified  $\mathcal{C}_j$ ,  $j = 1, \dots, 8$ , we can define the equivalent system  $(\Sigma_z)$  and construct the reachable set sequence (15) initialized at  $\{0\}$  by applying the map (16). By Proposition 2, the limit of this sequence exists, and the mRPI set, shown in Fig. 7 is recovered in 25 steps. In the same figure, we show in red the fixed point of the set sequence (17) reached in 100 steps, and is identical with the results in [13] utilising the projection mappings (10). We highlight, as zoomed in in Fig. 7, left, upper right corner, that the convex RPI set (red) is *not* the convex hull of the mRPI set (grey).

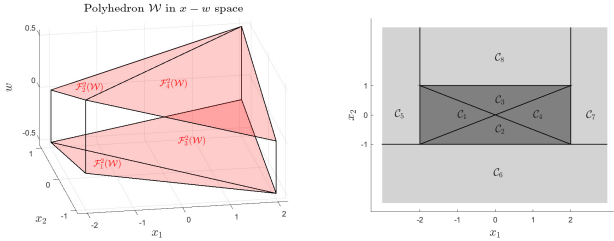


Fig. 5. Left: Polytope  $\mathcal{W}$  in the  $x-w$  space, with  $\mathcal{F}_i^2(\mathcal{W})$  in red satisfying Theorem 2i). Right: the corresponding partition in  $\mathbb{R}^2$ .

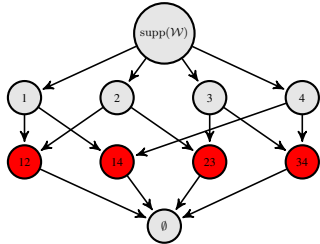


Fig. 6. The lattice  $L(P)$  induced by  $\mathcal{W}$  shown in Fig. 5 (left).

## VI. CONCLUSION

Linear systems subject to additive exogenous signals whose range is state dependent have been studied in the literature due to their significance and the challenges they pose. We showed for the first time they are equivalent to state

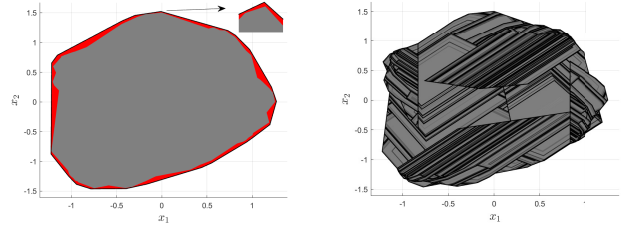


Fig. 7. Left: The (nonconvex) mRPI set (grey) and a convex RPI set in red color, generated by the sequences (15)-(16) and (17), respectively. Right: The union of the elements of the nonconvex mRPI set.

dependent piecewise affine polytopic systems. Exploiting the properties of a lattice induced by polytopic projections, we derived a complete partition of the state space defining this dynamics. Although piecewise affine dynamics typically give rise to complexity issues in reachability analysis, convexity in the support of the disturbance set suggests that this new outlook provides, other than a previously unseen connection of these systems to fundamental hybrid dynamics, a computationally appealing alternative.

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