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From relaxed constraint satisfaction to p -invariance of sets

Sorin Oлару, Martin Soyer, Zhixin Zhao, Carlos E. T. Dórea, Ernesto Kofman and Antoine Girard

Abstract—The present paper proposes a general framework for the analysis of constraints satisfaction with respect to the trajectories of a dynamical system. The results are presented in a discrete-time framework and pertain to the class of set-theoretic methods. The main objective is to go beyond the state-of-the-art by characterizing the intermittent constraint satisfaction along the evolution of the trajectories of a dynamical system. Two relaxed notions are introduced in this sense, one characterizing the validation of constraints within a given finite window and the other imposing the validation after a fixed number of time-steps following a violation. The constraint satisfaction with respect to a controlled trajectory will then be extended to a set of constraints and then to tubes of trajectories. It is shown that all these notions can be accordingly anchored to the well-known controlled positive set invariance, thus offering a generalized framework for the analysis of dynamical systems in a set-theoretic framework. The technical note is completed with illustrations of the constructions on both linear and nonlinear case.

I. INTRODUCTION

Constraints handling along the evolution of the trajectories of a dynamical system represented a continuous interest in the control literature [1]- [2] and extended recently to safety desiderata [3] with respect to the closed-loop functioning. Essentially, for constrained dynamical systems analysis, the goal is similar to the one of qualitative theory of differential equations: obtaining certificates of the constraints satisfaction along the trajectories without effectively building these trajectories. In this context, when accounting for any feasible initial conditions of the dynamics, the all-time constraints satisfaction lead naturally to the notion of positive invariance of the respective set of constraints [4].

Whenever it can be effectively characterized, the positive invariance is the tool of choice for the constraints satisfaction proofs. However, the rigid invariance conditions lead to complex geometrical shapes and active research is dedicated to the construction of positive invariant sets all by managing their complexity. This complexity remains the major issue in the case of high dimensional or nonlinear dynamics. To circumvent the problem, several authors point to relaxations, extensions or alternative notions. Some of these attempts are briefly described next:

- Periodic/Extended invariance [5] [6] impose the validation of the set-membership only periodically and

S. Oлару, M. Soyer, Z. Zhao and A. Girard are with Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France. (e-mails: martin.soyer; sorin.olaru; zhao.zhixin; antoine.girard@cenralesupelec.fr).

Carlos E. T. Dórea is with Department of Computer Engineering and Automation, UFRN. Natal, Brazil (e-mail: cetdorea@dca.ufrn.br).

Ernesto Kofman is with Departamento de Control, FCEIA, UNR. Rosario, Argentina (e-mail: kofman@fceia.unr.edu.ar).

thus provides a relaxation with respect to invariance conditions. However, the constrained satisfaction being imposed for any initial conditions within the constrained set and irrespective of the trajectory followed, it does not involve the history of constraint violation.

- Periodic invariant sequence [7], [8] similar to extended invariance, provides additionally a collection of (ordered) transition sets along the constraints' violation window.
- Cyclic invariance has been proposed in the context of time-delay systems [9] as a relaxation of the positive invariance through the use of a cyclic set-inclusions within a family of ordered sets.
- Invariant family of sets [10] is used in the context of decentralized systems to replace the classical invariance notion through the decomposition within a finite family of sets thus generalizing the periodic sequences of sets.
- Finite-time contractive sets pursue the development of periodic invariance notion by parameterizing the index of constraints' violation and associates it with a "contraction" factor [11], leading to the notion of (k, λ) -invariance .
- Confinement sets have been introduced [12] as a tool for the characterization of convex Lyapunov functions. Essentially, they are used to prove that convex sets are too restrictive to characterize the invariance for nonlinear (e.g. switched) systems.
- Probabilistic Invariance [13], [14] extends the concept of positive invariant sets in presence of unbounded disturbances such as white Gaussian Noise. If the initial state is inside a probabilistic invariant set, then every future state will belong to that set with a probability larger than certain parameter.
- Non-monotonic Lyapunov functions On a larger scope, if the invariant sets are interpreted as level sets of Lyapunov functions, the relaxation of invariance can be related to a relaxation of the monotonicity of the Lyapunov function [15] - [16] or even to cyclomonotonicity [17].

The common feature of these approaches is the relaxation of any-time set-membership satisfaction. However, a fundamental information is not taken into account in those existing notions: the starting points of the window of constraints violation. As such, the conditions imposed in the previous results are exclusively related to the initial conditions and do not account for the history of constraint satisfaction of the trajectory reaching the respective point in the state space (alternatively, they do not consider the trajectory of the

system as a whole). The objective of the present work is to offer a theoretical ground for the treatment of the intermittent constraints satisfaction along the trajectories of dynamical systems by using all these information. The ultimate goal remains the relaxation of the positive invariance conditions and the potential decrease of the complexity of the respective sets.

If the system is subject to multiple constraints, the classical set-invariance is inherently treating all of them concomitantly. When their satisfaction is relaxed, the analysis of the dynamics has to be dealt with at the individual level of each constraint to establish invariance conditions. This is an additional novelty in the analysis proposed by the present work.

The first contribution of the present paper is the description of the tools for the treatment of the intermittent satisfaction of constraints on a time-window of length p . This leads to the strong and weak notions of p -satisfaction of constraints according to their validation at exactly (strong) or at least (weak) p instants after the initialization of a sequence of violations. Subsequently, the p -satisfaction of constraints along the trajectories is extended for the tubes of trajectories.

The second contribution of the paper is the characterization of p -invariance by exploiting the p -satisfaction of a set of constraints for a tube of trajectory initiated within the set itself. The matching of initial conditions for the tube of trajectories p -satisfying some given constraints, to the set defined by the constraints per se, is essential in this construction.

It is worth mentioning that the framework proposed here for the p -invariance retrieves the classical notions whenever $p = 1$. The paper includes a section dedicated to effective construction of the p -invariant sets.

Basic Notations: In the following, the set of positive integers will be denoted \mathbb{N} . We will denote t_j as the j^{th} element of the ordered set $S = \{t_j\}_{j \in \mathbb{N}}$. We will denote by $\lceil a \rceil$ the closest integer of the real a . The subset of integers between a and b is denoted $\mathbb{N}_{[a,b]} = \mathbb{N} \cap [a, b]$. The Minkowski sum of two set A and B is denoted by $A \oplus B = \{a + b \mid a \in A, b \in B\}$.

II. INTERMITTENT CONSTRAINTS SATISFACTION FOR NONLINEAR SYSTEM'S TRAJECTORIES

Consider the generic discrete-time dynamical system:

$$x(t+1) = f(x(t), u(t)) \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m$ is single valued in both arguments. We will consider a given set of state constraints $X \subset \mathbb{R}^n$ and input constraints $U \subset \mathbb{R}^m$. The following problems are to be addressed:

- 1) Characterize the violation of the frontiers of X for a finite time-window of at most p instants by the solution of (1) with a given initial condition x_0 and an associated feasible control sequence.
- 2) Provide conditions to ensure that any trajectory of (1) initiated in X violates the set-membership constrained $x(t) \in X$ for at most p instants.

- 3) Characterize the satisfaction of the set-constraints $x(t) \in X$ at exactly p instants after any violation.
- 4) Provide conditions to ensure that any trajectory of (1) initiated in X , verifies the set-membership constrained $x(t) \in X$ at exactly p instants after any violation.

Since this work deals with a generalization of positive invariance, we recall here its formal definition:

Definition 1. A set Ω is controlled positively invariant with respect to (1) if the set-membership validation at current state $x(t) \in \Omega$ implies the existence of a control actions $u(t+i) \in U$ such that the state trajectory satisfies $x(t+i) \in \Omega, i \in \mathbb{N}$.

Even if the invariance is often use for constraint satisfaction guarantees, is obvious that such strong all-time constraint satisfaction condition does not provide the appropriate theoretical framework for addressing the problems above.

Notation: The t -th iteration ($t \in \mathbb{N}$) of the system (1) in closed-loop with a state feedback function $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is denoted by $f^t(x_0, u) = f(f^{t-1}(x_0, u), u(f^{t-1}(x_0, u)))$ and $f^0(x_0, u(x_0)) = x_0$. We denote the set mapping representing the one-step predecessor set of Ω with input restrictions U as: $Pre_f^1(\Omega) = \{x \in \mathbb{R}^n \mid \exists u \in U, f(x, u) \in \Omega\}$. We generalize this definition to t steps iteration: $Pre_f^t(\Omega) = Pre_f^{t-1}(Pre_f(\Omega))$ and $Pre_f^0(\Omega) = \Omega$.

A. Validation Index Set

Consider the model (1) and a continuous function $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$. We define the sublevel set of h as follows¹:

$$\mathcal{L}(h) = \{x \in \mathbb{R}^n, u \in \mathbb{R}^m \mid h(x, u) \leq 0\} \quad (2)$$

Assumption II.1. h is such that $\mathcal{L}(h)$ is a closed set of \mathbb{R}^n .

The main goal being to study the membership of a trajectory to the sublevel set $\mathcal{L}(h)$, the next definition is introduced.

Definition 2. Given an initial state $x_0 \in \mathbb{R}^n$ and an admissible state-feedback control function $u : \mathbb{R}^n \rightarrow U$, the *Validation Index Set (VIS)* $\mathcal{V}(x_0, f, h, u) \subset \mathbb{N}$ is an ordered collection of indices for which the states of (1) belong to the sublevel set $\mathcal{L}(h)$.

Definition 3. The *maximal Validation Index Set (mVIS)* is the maximal ordered collection of indices VIS (w.r.t set inclusion). It can be formally described as:

$$\mathcal{V}^m(x_0, f, h, u) = \left\{ t \in \mathbb{N} \left| \begin{array}{l} x(t) \in \mathcal{L}(h); x(0) = x_0 \\ x(t+1) = f(x(t), u(x(t))) \end{array} \right. \right\} \quad (3)$$

With the above definitions, a series of basic properties can be inferred:

Properties II.1. For a system (1), the following holds:

- If $\bar{u} \in U$ such that

$$\bar{x} = f(\bar{x}, \bar{u})$$

¹In this work we will refer indifferently to the satisfaction of the constraint $h(x, u) \leq 0$ and the inclusion $(x, u) \in \mathcal{L}(h)$ as the *satisfaction* of $\mathcal{L}(h)$.

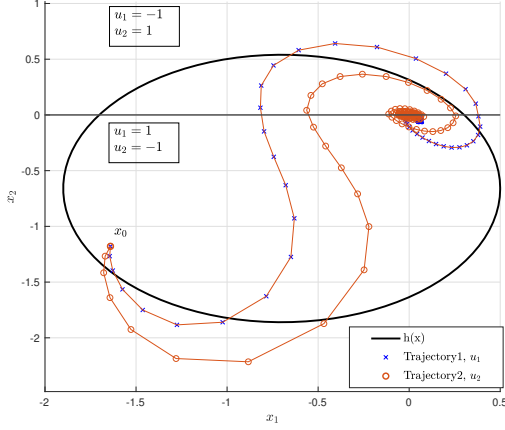


Fig. 1: The state trajectory validates intermittently the constraints.

then either $\mathcal{V}(\bar{x}, f, h, \bar{u}) = \mathbb{N}$ or $\mathcal{V}(\bar{x}, f, h, \bar{u}) = \emptyset$.

- Let a state-feedback function $u : \mathbb{R}^n \rightarrow U$. For all $t \in \mathbb{N}$, $t \oplus \mathcal{V}^m(f^t(x_0, u), f, h, u)$ is a VIS and the following inclusion holds:

$$t \oplus \mathcal{V}^m(f^t(x_0, u), f, h, u) \subset \mathcal{V}^m(x_0, f, h, u).$$

Example II.1. Consider the dynamical nonlinear system:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_1(t) + 0.25x_2(t) + 0.2x_2^2(t) + 0.0125u(t) \\ -0.25x_1(t) + 0.75x_2(t) + 0.25x_1^2(t)x_2(t) \end{bmatrix}$$

Consider the sublevel set corresponding to the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that:

$$h(x) = (x_1 + 0.7)^2 + (x_2 + 0.66)^2 - 1.44$$

Then $\mathcal{L}(h)$ depicts the circle of radius 1.2 and trajectory is initiated at $x_0 \triangleq x(0) = [-1.64, -1.18]^T$. They differ with respect to the control input functions:

$$u_1(x) = \begin{cases} 1 & \text{if } x_2 \leq 0 \\ -1 & \text{if } x_2 > 0 \end{cases}, u_2(x) = \begin{cases} 1 & \text{if } x_2 \geq 0 \\ -1 & \text{if } x_2 < 0 \end{cases}$$

Fig.1 depicts the configuration and we can construct the mVIS for the 2 cases: $\mathcal{V}^m(x_0, f, h, u_1) = \{0, 1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 25, \dots\}$ and the counterpart $\mathcal{V}^m(x_0, f, h, u_2) = \{0, 1, 8, \dots\}$.

Assuming a dynamic system (1) and a constraint modeled by a sublevel set $\mathcal{L}(h)$ as defined by (2), we now focus on the case a trajectory violates the constraint for a finite time interval before satisfying it again.

B. Weak p -satisfaction of a constraint

Definition 4 (Weak p -satisfaction of constraint). Given finite $p \in \mathbb{N}_{>0}$, the trajectory of the system (1) initiated in $x_0 \in \mathbb{R}^n$ and subject to the state-feedback control $u : \mathbb{R}^n \rightarrow U$ is weakly p -satisfying the constraint $\mathcal{L}(h)$ if it exists a function $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}$ such that $x(t+r(t)) \in \mathcal{L}(h)$ for any $t \in \mathbb{N}$.

In other words, a trajectory weakly p -satisfies a constraint if it reaches $\mathcal{L}(h)$ at least once during any time interval of

length p . We provide a necessary and sufficient condition of weak p -satisfaction of a constraint with respect to a trajectory.

Theorem II.1. Given a finite $p \in \mathbb{N}_{>0}$, the trajectory of the system (1) initiated in x_0 , and subject to the state-feedback control $u : \mathbb{R}^n \rightarrow U$ weakly p -satisfies the constraint $\mathcal{L}(h)$ if and only if :

- the mVIS $\mathcal{V}^m(x_0, f, h, u)$ is unbounded
- The difference between two consecutive elements is bounded by:

$$\begin{aligned} \sigma &= \max_{j \in \mathbb{N}} t_{j+1} - t_j \\ \text{s.t. } & (t_j, t_{j+1}) \in \mathcal{V}^m(x_0, f, h, u) \end{aligned} \quad (4)$$

- This bound satisfies the condition $\sigma \leq p$.

Proof. If the trajectory weakly p -satisfies $\mathcal{L}(h)$ then there exists a function $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}$ such that

$$x(t+r(t)) \in \mathcal{L}(h) \quad \forall t \in \mathbb{N}$$

and the mVIS verifies:

$$\mathcal{S} = \{r(0), r(0) + r(r(0)), r(0) + r^2(0) + r(r(0) + r^2(0)), \dots\} \subset \mathcal{V}^m(x_0, f, h, u)$$

\mathcal{S} being an nonempty and unbounded subset of $\mathcal{V}^m(x_0, f, h, u)$, the optimization problem is feasible (any element $t_j \in \mathcal{V}^m(x_0, f, h, u)$ has a successor $t_{j+1} \in \mathcal{V}^m(x_0, f, h, u)$ in this ordered set). Conversely, one of the following relationships holds: $\mathcal{V}^m(x_0, f, h, u) = \mathcal{S}$ or $\mathcal{V}^m(x_0, f, h, u) = \{0, \mathcal{S}\}$. In short,

$$\mathcal{V}^m(x_0, f, h, u) \subseteq \{0, \mathcal{S}\}. \quad (5)$$

Consider any pair of successive points $(t_j, t_{j+1}) \in \mathcal{V}^m(x_0, f, h, u) \times \mathcal{V}^m(x_0, f, h, u)$ with $t_j \geq r(0)$. Using (5), it exists a pair of successive points $(s_i, s_{i+1}) \in \mathcal{S} \times \mathcal{S}$ such that $s_i \leq t_j \leq t_{j+1} \leq s_{i+1}$.

$$\begin{aligned} t_{j+1} - t_j &\leq s_{i+1} - s_i \\ &\leq r(s_i) \leq p \end{aligned}$$

For any pair of successive points $(t_j, t_{j+1}) \in \mathcal{V}^m(x_0, f, h, u) \times \mathcal{V}^m(x_0, f, h, u)$ with $t_j < r(0)$ we have $t_{j+1} \leq r(0)$ because $\mathcal{V}^m(x_0, f, h, u)$ is the maximal VIS. It follows that:

$$t_{j+1} - t_j \leq r(0) - t_j \leq r(0) \leq p$$

For the sufficiency, one can note that $\mathcal{V}^m(x_0, f, h, u) = (t_i)_{i \in \mathbb{N}}$ being (ordered) unbounded set then one can define the function :

$$\begin{aligned} r : \mathbb{N} &\rightarrow \mathbb{N}_{[1,p]} \\ r(t) &= \begin{cases} t_0 - t & \text{if } t < t_0 \\ t_{j+1} - t & \text{if } t_j \leq t < t_{j+1} \end{cases} \end{aligned}$$

thus $t+r(t) \in \mathcal{V}^m(x_0, f, h, u)$ and the trajectory $(x(t))_{t \in \mathbb{N}}$ is weakly p -satisfying the constraint according to Def. 4. \square

In other words, a trajectory weakly p -satisfies a constraint if and only if two successive elements of the mVIS are

separated by a interval bounded by p . The notion can be extended naturally to the controlled setting through the next definition.

Definition 5 (Weak controlled p -satisfaction of constraint). *Given finite $p \in \mathbb{N}_{>0}$, the trajectory of the system (1) initiated in $x_0 \in \mathbb{R}^n$ is weakly controlled p -satisfying the constraint $\mathcal{L}(h)$ if it exist a state feedback control function $u : \mathbb{R}^n \rightarrow U$ and $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}$ such that $x(t + r(t)) \in \mathcal{L}(h)$ for any $t \in \mathbb{N}$.*

C. Weak p -satisfaction of a vector-constraint

Notions of VIS and weak p -satisfaction can be extended to vector constraints $h^l : \mathbb{R}^n \rightarrow \mathbb{R}^l$, where $h^l = [h_1^l, \dots, h_l^l]^\top$. For a compact presentation, the controlled notions will be discussed in the remainder.

Definition 6 (Weak vector-constraint satisfaction). *A controlled trajectory of the system (1) initiated in $x_0 \in \mathbb{R}^n$ weakly p -satisfies the vector constraint $\mathcal{L}(h^l)$ if it exists $u : \mathbb{R}^n \rightarrow U$ and an associated $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}^m$ such that*

$$x(t + r_i(t)) \in \mathcal{L}(h_i^l), \forall i \in \{1, \dots, l\}$$

for any $t \in \mathbb{N}$ where $r_i(\cdot)$ is the i -th component of r .

Theorem II.2. *The trajectory of the system (1) with an initial condition $x_0 \in \mathbb{R}^n$ weakly satisfies the m -dimensional vector-constraint $\mathcal{L}(h^l)$ if and only if it exists $u : \mathbb{R}^n \rightarrow U$ such that*

- All the mVIS $\mathcal{V}^m(x_0, f, h_i^l, u)$ with $i \in \mathbb{N}_{[1,l]}$ are unbounded;
- The next optimization problem has a bounded solution:

$$\begin{aligned} \max_{i \in \mathbb{N}_{[1,l]}} \quad & \max_{j \in \mathbb{N}} \{t_{j+1} - t_j\} \\ \text{s.t.} \quad & (t_j, t_{j+1}) \in \mathcal{V}^m(x_0, f, h_i^l, u). \end{aligned} \quad (6)$$

Proof. According to Theorem II.1 applied here for each $i \in \mathbb{N}_{[1,l]}$, the constraint $\mathcal{L}(h_i^l)$ is weakly satisfied by the controlled trajectory if and only if the optimization problem (4) has bounded solution.

As a consequence, the optimization (6) is feasible and has a bounded solution given by the most conservative index of weak satisfaction. \square

Remark 1. *The weak satisfaction of the multiple constraints does not guarantee the simultaneous satisfaction of constraints on a finite time interval despite the global weak-satisfaction of the vector-constraint.*

The next step is to extend these notions for tubes of trajectories as shown next.

Definition 7. (Controlled weak p -satisfaction of constraint for a tube of trajectories) *Given $p \in \mathbb{N}_{>0}$, the controlled trajectories of the system (1) initiated in $X \subset \mathbb{R}^n$ weakly p -satisfy the vector constraint $\mathcal{L}(h^l)$ if it exists $u : \mathbb{R}^n \rightarrow U$ and an associated function $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}^m$ such that*

$$x(t + r_i(t)) \in \mathcal{L}(h_i^l), \forall i \in \{1, \dots, l\}$$

for all $x_0 \in X$ and for any $t \in \mathbb{N}$ where $r_i(\cdot)$ is the i -th component of r .

Theorem II.3. *The tube of trajectories of the system (1) initiated in $X \subset \mathbb{R}^n$ weakly p -satisfies the l -dimensional vector-constraint $\mathcal{L}(h^l)$ if and only if it exists a state feedback control $u : \mathbb{R}^n \rightarrow U$ ensuring*

- All the mVIS $\mathcal{V}^m(X, f, h_i^l, u)$ with $i \in \mathbb{N}_{[1,l]}$ are unbounded;
- The next optimization problem has a bounded solution:

$$\begin{aligned} \max_{x \in X, i \in \mathbb{N}_{[1,l]}} \quad & \max_{j \in \mathbb{N}} \{t_{j+1} - t_j\} \\ \text{s.t.} \quad & (t_j, t_{j+1}) \in \mathcal{V}^m(x, f, h_i^l, u) \end{aligned} \quad (7)$$

Proof. Theorem II.3 is verified if and only if every trajectory initialised in X weakly p -satisfy the vector-constraint $\mathcal{L}(h^l)$ and consequently if and only if every trajectory verifies Theorem II.2. \square

Up to this point, the set of initial conditions $X \subset \mathbb{R}^n$ was considered to be independent of the constraints whose satisfaction is under study i.e. $h^l(x) \leq 0$. In the case the set is defined as $X = \mathcal{L}(h^l)$ one can talk about the self p -satisfaction of constraints and further about p -invariance, as described in Section III.

III. WEAK SET INVARIANCE

In the previous section we defined notions of p -satisfaction of constraints for trajectories and tubes of trajectories and we will extend those properties to the definition of p -invariance for a set. A set can be defined as an intersection of constraints and the weak p -satisfaction of each such constraint can be analysed in the previous framework. If this analysis is carried out with respect to initial states in the set itself we talk about weak notion of p -invariance of a set.

Definition 8 (Controlled weak invariance). *Let $p \in \mathbb{N}$. The set $\Omega \subset \mathbb{R}^n$ is controlled weakly p -invariant with respect to the system (1) if it exists a state feedback control function $u : \mathbb{R}^n \rightarrow U$ such that for any $x_0 \in \Omega$, one can associate an index function $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}$ satisfying $x(t + r(t)) \in \Omega$ for any $t \in \mathbb{N}$.*

Proposition III.1. *A set $\Omega \subset \mathbb{R}^n$ is controlled weakly p -invariant with respect to the system (1) if and only if the constraint $h_\Omega(x) \leq 0$ defined by the function:*

$$\begin{aligned} h_\Omega : \mathbb{R}^n & \longrightarrow \mathbb{R} \\ h_\Omega(x) & = \begin{cases} 1 & \text{if } x \notin \Omega \\ -1 & \text{if } x \in \Omega \end{cases} \end{aligned} \quad (8)$$

is controlled weakly p -satisfied by the tube of trajectories initiated in Ω .

The latter proposition has only a theoretical interest and it can be complemented with the following properties to be used in practical cases.

Theorem III.1. *A set $\Omega \subset \mathbb{R}^n$ is controlled weakly p -invariant with respect to the system (1) if and only if it exists a state feedback control function $u : \mathbb{R}^n \rightarrow U$ such that*

- $\mathcal{V}^m(x_0, f, h_\Omega, u)$ is unbounded for all $x_0 \in \Omega$
- The optimization problem:

$$\begin{aligned} \sigma &= \max_{x_0 \in \Omega} \max_{j \in \mathbb{N}} \{t_{j+1} - t_j\} \\ \text{s.t.} \quad & (t_j, t_{j+1}) \in \mathcal{V}^m(x_0, f, h_\Omega, u) \end{aligned} \quad (9)$$

has a bounded solution.

- $\sigma \leq p$

Proof. If the set Ω is controlled weakly p -invariant, it exists a $u(\cdot)$ such that the unboundedness of $\mathcal{V}^m(x_0, f, h, u)$ is verified for every $x_0 \in \Omega$ following the same arguments as Theorem II.1. Consequently, the optimization problem is feasible and its solution is bounded by p thanks to the upper bound on the validation index function in Definition 7.

Conversely, the satisfaction of the first conditions proves the feasibility of the optimization problem. By choosing p as feasible solution in the optimization, the conditions of weak invariance are fulfilled. \square

The following two corollaries are direct applications of the definition and the theorem above.

Corollary III.2. *If a set $\Omega \subset \mathbb{R}^n$ is controlled weakly p_0 -invariant with respect to the system (1) then it will be controlled weakly p -invariant for all $p > p_0$.*

Corollary III.3. *The controlled weakly 1-invariance is equivalent to the positive invariance (Definiton 1).*

The set Ω has been used up to this point based on the associated indicator function h_Ω in (III.1) but its description can be done explicitly based on joint satisfaction of a set of constraints:

$$\Omega = \bigcap_{i=1}^l \mathcal{L}(h_i). \quad (10)$$

A link between these two notions can be established.

Proposition III.2. *If the set Ω defined as an intersection of constraints (10) is controlled weakly p -invariant, then the tube of controlled trajectories initiated in Ω weakly p -satisfies the constraints $\mathcal{L}(h_i) \forall i$ (or equivalently the vector-constraint $h = [h_1, \dots, h_l]^T$).*

The proof is a direct consequence of the fact that $h_i(x) \leq 0$ for all x satisfying $h_\Omega(x) \leq 0$. Is important is to note that the converse is not true as exemplified graphically by the Fig. 2.

Corollary III.4. *Let the set Ω defined as an intersection of constraints (10). If it exists $p \in \mathbb{N}$ such that Ω is controlled weakly p -invariant, then the constraints $\mathcal{L}(h_i)$ are controlled weakly p_i -satisfied with*

$$p \geq \max_{i=1 \dots l} p_i \quad (11)$$

Theorem III.5. Ω is controlled weakly p -invariant with respect to (1) if and only if:

$$\Omega \subseteq \bigcup_{i=1}^p \text{Pre}_f^i(\Omega). \quad (12)$$

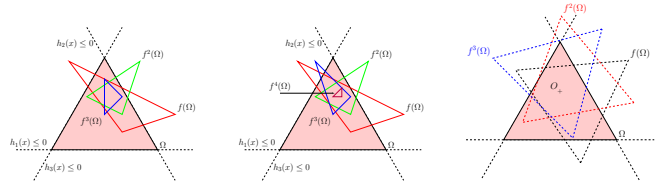


Fig. 2: Left: $p = \max_i p_i$. Middle: $p > \max_i p_i$. Right: no weak p -invariance while every constraints are weakly p -satisfied

Proof. (IF) Condition (12) implies that, $\forall t \in \mathbb{N}$, if $x(t) \in \Omega$, then there exists $u : \mathbb{R}^n \rightarrow U$ such that $x(t) \in \bigcup_{i=1}^p \text{Pre}_f^i(\Omega)$ and thus there exists $1 \leq r(t) \leq p$ to ensure $x(t+r(t)) \in \Omega$. Since it is also true for $t = 0$, then, for any $x_0 = x(0) \in \Omega$, there exists $r : \mathbb{N} \rightarrow \mathbb{N}_{[1,p]}$ such that $x(t+r(t)) \in \Omega$ which proves the sufficiency.

(ONLY IF) By contradiction, consider $x(0) = x_0 \in \Omega$, but $x_0 \notin \bigcup_{i=1}^p \text{Pre}_f^i(\Omega)$. Then, for the controlled trajectory on the finite window is such that $x(i) \notin \Omega \forall i = 1, \dots, p$, and thus Ω does not satisfy the conditions of weak p -invariance. \square

IV. STRONG NOTIONS OF p -SATISFACTION OF CONSTRAINTS AND p -INVARIANCE

The weak constraint satisfaction and weak invariance offer two different perspectives on the validation of static constraints along the trajectories of a dynamical system. A strong version will enhance the framework by imposing additional restrictions on the allowed interval between violation of constraints. More specifically, the validation of constraints has to be guaranteed at exactly p instants after any violation.

Definition 9 (Controlled strong p -satisfaction of vector-constraints). *The tube of trajectories of the system (1) initialized in $X \subset \mathbb{R}^n$ is strongly p -satisfying the vector constraints $\mathcal{L}(h)$ with $h : \mathbb{R}^n \rightarrow \mathbb{R}^l$ if it exists a state feedback control function $u : \mathbb{R}^n \rightarrow U$ such that in closed-loop:*

$$x(t+p) \in \mathcal{L}(h_i) \forall i \in \mathbb{N}_{[1,l]},$$

for any $x_0 \in X$ and for any time instant $t \in \mathbb{N}$ for which $x(t) \in \mathcal{L}(h_i)$ and $x(t+1) \notin \mathcal{L}(h_i)$.

Theorem IV.1. *The l -dimensional vector-constraint $h(x) \leq 0$ is strongly p -satisfied by the tube of trajectories of the system (1) with initial conditions in $X \subset \mathbb{R}^n$ if and only if it exists a state feedback control $u : \mathbb{R}^n \rightarrow U$ ensuring $\forall x_0 \in X$*

- The sets $\mathcal{V}(x_0, f, h_i, u)$ with $i \in \mathbb{N}_{[1,l]}$ are unbounded;
- For each $i \in \mathbb{N}_{[1,l]}$ and for any successive elements $t_j \in \mathcal{V}(x_0, f, h_i, u)$ we have either $t_j + 1 \in \mathcal{V}(x_0, f, h_i, u)$ or $t_j + p \in \mathcal{V}(x_0, f, h_i, u)$.

Proof. This is a direct application of the definition through the alternatives with respect to the consecutive validation indices which are distanced by 1 or p . \square

Before establishing further results, we can notice that strong constrained satisfaction imposes stronger limitations

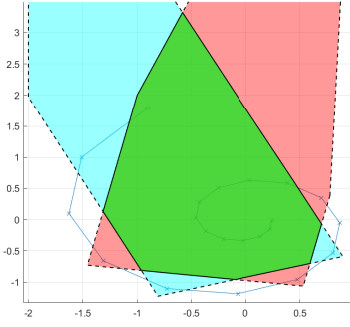


Fig. 3: The trajectory strongly 3-satisfies the blue constraints as well as the red ones but it does not strongly 3-satisfies their intersection (in green)

on the set of validation indices and thus is expected to lead to larger values of p with respect to the weak counterpart. The next result stresses that strong notion covers the weak counterpart.

Proposition IV.1. *If the l -dimensional vector-constraint $h(x) \leq 0$ is strongly p -satisfied by the trajectory of the system (1) with initial condition $x_0 \in X \subset \mathbb{R}^n$, then the same vector-constraint is weakly p -satisfied by the same trajectory.*

Proof. By choosing $r(t) = p, \forall t$ one retrieves Definition 8. \square

Remark 2. *Given Ω_1 and Ω_2 , the strong p -satisfaction of constraints $h_{\Omega_1} \leq 0$ and $h_{\Omega_2} \leq 0$ does not imply the strong p -satisfaction of $h_{\Omega_1 \cap \Omega_2}$ as it is illustrated in Fig. 3.*

Definition 10 (Controlled strong invariance). *Let $p \in \mathbb{N}$. The set $\Omega \subset \mathbb{R}^n$ is controlled strongly p -invariant with respect to the system (1) if it exists a state feedback control function $u : \mathbb{R}^n \rightarrow U$ such that, in closed-loop, $x(t+p) \in \Omega$ for each $x_0 \in \Omega$, and for any $t \in \mathbb{N}$ for which $x(t) \in \Omega$ and $x(t+1) \notin \Omega$.*

Proposition IV.2. *If Ω is a controlled strongly p -invariant set, then it is also a controlled kp invariant set for all $k \in \mathbb{N}_{>0}$.*

Proof. It follows from Definition 10 where the controlled strongly p -invariant set ensures the trajectory initiated in the set are within the set at any multiple of p time indices. \square

Theorem IV.2. *A set $\Omega \subset \mathbb{R}^n$ is controlled strongly p -invariant with respect to the system (1) if and only if*

$$\Omega \subseteq \text{Pre}_f^p(\Omega) \cup \text{Pre}_f^1(\Omega). \quad (13)$$

Proof. (IF) follows from the Proposition IV.2.

(ONLY IF) The relationship (13) places the controlled strong p -invariance property in the set-theoretic framework. For any initial condition $x_0 \in \Omega$ two possibilities appear:

- $x \in \text{Pre}_f^1(\Omega)$

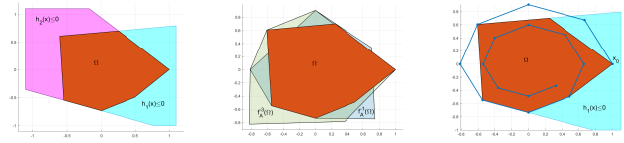


Fig. 4: Left: $\Omega = \mathcal{L}(h_1) \cap \mathcal{L}(h_2)$. Center: Graphical proof for the satisfaction of the condition (13) of 3-strong invariance of Ω for the closed loop trajectory of (1) representing a linear dynamics with rotation $\theta = \pi/4$ and contraction factor $\rho = 0.95$. Right: trajectory initiated in $\mathcal{L}(h_1)$ which violates the strong 3-satisfaction of constraints.

- $x \notin \text{Pre}_f^1(\Omega)$ and from (13) it follows that $x \in \text{Pre}_f^p(\Omega)$. Thus there exists a control sequence $u(k), k \in \mathbb{N}_{[0, p-1]}$ such that $x(p) \in \Omega$ with $x(k+1) = f(x(k), u(k)), k \in \mathbb{N}_{[0, p-1]}$ and $x(0) = x_0$. \square

Corollary IV.3. *If the set $\Omega \subset \mathbb{R}^n$ is controlled strongly p -invariant with respect to the system (1) then it is controlled weakly p -invariant.*

Proof. Analogous to Proposition IV.1. \square

It follows also from the same result that strong invariance index is greater than or equal to the weak invariance index. Next result bridges the gap between the strong p -satisfaction of (self)-constraints and the strong positive p -invariance.

Proposition IV.3. *Let a set Ω defined as in (10). If the set is controlled strongly p -invariant with respect to the dynamics (1) then the trajectories initiated in Ω are controlled weakly p -satisfying the vector-constraints defining this set.*

Proof. If Ω is controlled strongly p -invariant, then we have:

$$f^p(x, u(x)) \in \Omega \text{ for all } x \text{ s.t. } x \in \Omega \text{ and } f(x, u(x)) \notin \Omega$$

As $\Omega = \bigcap_{i=1}^l \mathcal{L}(h_i)$, we have for all $i \in \{1, \dots, l\}$:

$$f^p(x, u(x)) \in \mathcal{L}(h_i) \text{ for all } x \text{ s.t. } x \in \Omega \text{ and } f(x, u(x)) \notin \Omega$$

By choosing:

$$r_i(x) = \begin{cases} p & \text{if } x \in \Omega, f(x, u(x)) \notin \Omega \\ 1 & \text{if } x \in \Omega, f(x, u(x)) \in \Omega \\ 1 & \text{if } x \notin \Omega, x \in \mathcal{L}(h_i), f(x, u(x)) \in \mathcal{L}(h_i) \\ p-t & \text{if } x \notin \Omega, x \in \mathcal{L}(h_i), f(x, u(x)) \notin \mathcal{L}(h_i) \end{cases}$$

where $t < p$ is such that it exists $z \in \Omega$ such that $x = f^t(z, u(z))$. Thus, for any initial state $x \in \Omega$ it exists $r : \mathbb{N} \rightarrow \mathbb{N}_{[1, p]}$ such that $f^{r_i(x)}(x, u(x)) \in \mathcal{L}(h_i)$ which completes the proof. \square

Remark 3. *Strong p -invariance of Ω is not necessarily implying the strong p -satisfaction of constraints for the trajectories with initial conditions in Ω . Numerical examples in this sense can be constructed readily, as for example in Figure 4.*

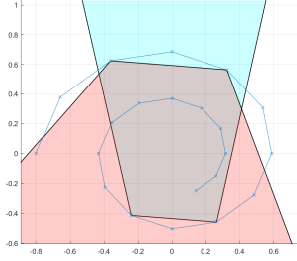


Fig. 5: Counter-example for the converse implication with respect to the statement of the Proposition IV.3

An interesting fact concerns the converse implications, we show by a numerical example that neither the weak p -satisfaction of constraints nor their strong p -satisfaction will imply the strong p -invariance.

Counter-example: Consider the system

$$x(t+1) = \rho \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} x(t), \text{ with } \rho = 0.95, \theta = -\frac{\pi}{6}$$

and two polyhedral constraints sets represented in blue and red regions in Fig.5 such that Ω is defined by their intersection. The constraints are weakly satisfied with an index $p = 4$ for the initial conditions in Ω . However, the set Ω is not strongly p -invariant with the same index.

Before concluding this section dedicated to the comparison between strong/weak p -invariance notions, let us recall the periodic invariance properties (see for example [5], [6]) in order to allow a comparison of the concepts introduced here.

Definition 11. A set Ω is said to be controlled p -periodic invariant with respect to (1) if :

$$\Omega \subseteq \text{Pre}_f^p(\Omega) \quad (14)$$

In the light of this definition, it becomes clear that a set which enjoys the periodic invariance property is strongly/weakly invariant and conversely that a strongly/weakly p -invariant set is not necessarily periodic invariant with the same index p .

Proposition IV.4. If a set Ω is controlled p -periodic invariant w.r.t (1) then it is controlled strongly (and weakly) p -invariant.

Proof. By exploiting the conditions (13), (12) and (14):

$$\Omega \subseteq \text{Pre}_f^p(\Omega) \subseteq \underbrace{\text{Pre}_f^p(\Omega)}_{\text{strong } p\text{-invariant}} \cup \underbrace{\text{Pre}_f^1(\Omega)}_{\text{weak } p\text{-invariant}} \subseteq \bigcup_{i=1}^p \text{Pre}_f^i(\Omega)$$

which concludes the proof. \square

V. CONSTRUCTION OF A CONTROLLED WEAKLY/STRONGLY p -INVARIANT SET

Constructing controlled p -invariant sets in the case of linear time-invariant systems

$$x(t+1) = f_{in}(x(t), u(t)) = Ax(t) + Bu(t)$$

can be done in a straightforward manner as long as the conditions (13) or (12) are based on the operator

$$\text{Pre}_{f_{in}}^1(\Omega) = \{x | \exists u \in U, \text{ s.t. } Ax + Bu \in \Omega\}$$

finding versatile implementation e.g. for polyhedral sets U, Ω .

The construction of a controlled weakly/strongly p -invariant set for a nonlinear systems (1) is difficult in a direct set-theoretic framework but can rely on well-established tools from the realm of symbolic control approach.

Symbolic control involves discretizing both the state space and the control input. It represents the set of states as a finite collection of symbols that transition among each other, influenced by the control input. This approach allows us to over-approximate a continuous model by a transition system $S = (Q, P, F)$, where Q denotes the set of symbolic states, P signifies the set of symbolic inputs, and $F : Q \times P \rightarrow 2^Q$ represents the set-valued transition map. If Q and P are finite, S is termed finite or symbolic [18].

Using this framework, a set of states $\Omega \subseteq \mathbb{R}^n$ is represented by its symbolic counterpart $Q_\Omega \subseteq Q$. Then one can compute a symbolic representation of the set $\text{Pre}_f^1(\Omega)$, defined as:

$$\text{Pre}_F^1(Q_\Omega) = \{q \in Q | \exists p \in P, \emptyset \neq F(q, p) \subseteq Q_\Omega\}$$

Given a candidate set Ω for the controlled weak/strong p -invariance, by progressively increasing the value of p until the verification of the relation (12) and (13), the set Ω can be certified as a controlled weakly/strongly p -invariant set for the transition system S and these properties will hold for the continuous (state vector-space) model.

The only issue in such construction is the finite determinedness. In short (see [19]), any candidate set Ω within the maximal control invariant set guarantees the existence of a finite index p . It should be recalled that the maximal control invariant set for the transition system can be also computed using a fixed-point algorithm [20].

Let us consider an example of controlled Hénon map:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.44 - 0.1x_3(t) - 4x_2(t)^2 + 0.25u \\ x_1(t) - 4x_1(t)x_2(t) \\ x_2(t) \end{bmatrix}$$

with the state constraint set $X = [-1, 1]^3$ and input constraint set $U = [-1, 1]$. Figure 6 depicts a computed controlled weak/strong p -invariant set. This computed set exhibits control properties akin to the control invariant set but has a more simplified representation, thereby allowing for potential substitution of the control invariant set in certain scenarios, for example in NMPC. Figure 7 shows the clear complexity attractiveness of the p -invariant set description (polytopic set with minimal half-space representation).

VI. DISCUSSION AND FURTHER WORK

Let us recall by means of a schematic representation, Figure 8, the different concepts developed in this paper and additionally summarize the links between them. The strong p -invariance is the most conservative among the set-theoretic

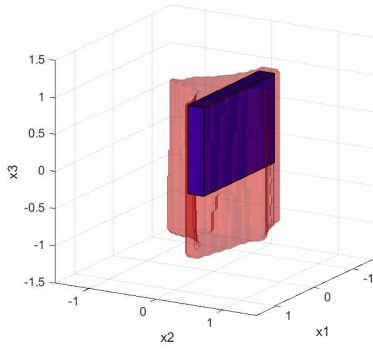


Fig. 6: The maximal control invariant set (in red) alongside a hyperrectangle (in blue), certified either as a controlled weakly 7-invariant set or as a controlled strongly 10-invariant set.

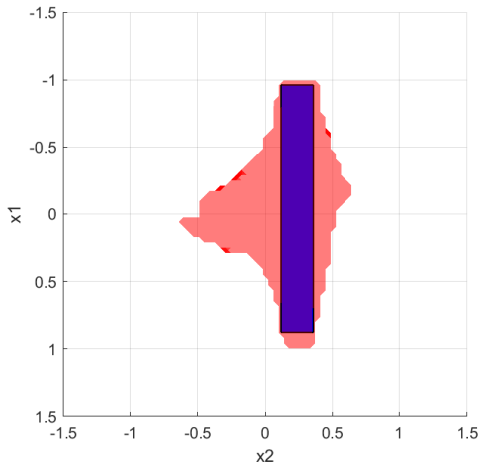


Fig. 7: Projection of the p -invariant set (blue) onto the first two dimensions showing the reduced complexity with respect to the one of the nonconvex controlled invariant set in red (note the different red colour intensity according to the depth on the third dimension).

properties and implies the weak p -invariance (Corollary IV.3, the converse is not true) and the weak p -satisfaction of constraints of a tube of trajectories (Proposition IV.3) which both imply the weak p -satisfaction of constraints (Proposition III.2 and Definition 7).

A new class of properties with respect to the constrained evolution of dynamical systems has been introduced and further works should be dedicated to the characterization of p -invariance with respect to operators such as intersection or union and define new parameterizations in terms of the index p . It would be also relevant to understand the inherent relationship between the p -invariance index and the complexity of the set representation. A particularly interesting question would be to find the less complex set corresponding to a given index p .

For the control design application, the use of p -invariance shows to be an important tool as already documented in [21] or [22]- [23].

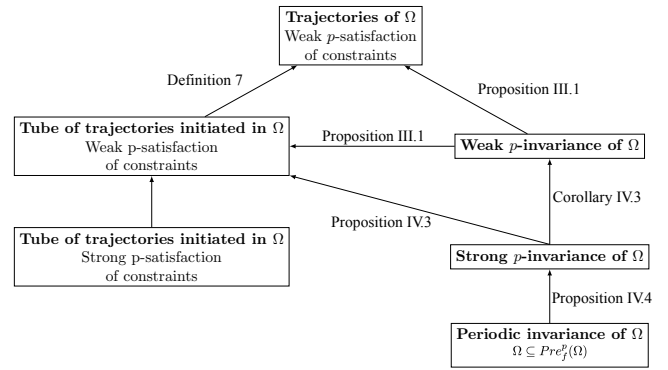


Fig. 8: Summary of the different notions of p -satisfaction and p -invariance. Arrows denote implication relations.

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