

Supplement to “Existence of a periodic and seasonal INAR process”

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A Proofs of the auxiliary results

Proof of equations (4). For each $\mathbf{n} \in \mathbb{N}_0^S$, we have that $\{\mathbf{Z} = \mathbf{n}\} \in \mathcal{F}$, $\mathbf{E}(M \circ \mathbf{n}) = M\mathbf{n}$ and $\text{Var}(M \circ \mathbf{n}) = \text{diag}(V\mathbf{n})$. Indeed,

$$\mathbf{E} \left(\sum_{j=1}^S m_{i,j} \circ n_j \right) = \sum_{j=1}^S \mathbf{E}(m_{i,j} \circ n_j) = \sum_{j=1}^S m_{i,j} n_j = (M\mathbf{n})_i$$

and, since the counting sequences involved into $M \circ$ are mutually independent, we have

$$\begin{aligned} \text{Cov} \left(\sum_{j=1}^S m_{i,j} \circ n_j, \sum_{l=1}^S m_{k,l} \circ n_l \right) &= \sum_{j=1}^S \sum_{l=1}^S \text{Cov}(m_{i,j} \circ n_j, m_{k,l} \circ n_l) \\ &= \sum_{j=1}^S \delta_{ik} \text{Var}(m_{i,j} \circ n_j) = \delta_{ik} \sum_{j=1}^S m_{i,j} (1 - m_{i,j}) n_j \\ &= \delta_{ik} \sum_{j=1}^S v_{i,j} n_j = \delta_{ik} (V\mathbf{n})_i \end{aligned}$$

for all $i, k = 1, \dots, S$. Then (4) follows by a truncation argument. Namely, we have that

$$\begin{aligned} \mathbf{E}(M \circ \mathbf{Z} \chi(\|\mathbf{Z}\| \leq n) | \mathcal{F}) &= \sum_{\|\mathbf{n}\| \leq n} \chi(\mathbf{Z} = \mathbf{n}) \mathbf{E}(M \circ \mathbf{n}) \\ &= \sum_{\|\mathbf{n}\| \leq n} \chi(\mathbf{Z} = \mathbf{n}) M\mathbf{n} = M\mathbf{Z} \chi(\|\mathbf{Z}\| \leq n) \end{aligned}$$

for all $n \in \mathbb{N}$, where $\chi(A)$ denotes the indicator function of a set A . By the monotone convergence theorem, this formula implies the first equality of (4) as $n \rightarrow \infty$. The proof of the second equality is similar. \square

Proof of Equation (10). Define the matrices $C_s = \sum_{j=2}^s \alpha_j \mathbf{e}_j \mathbf{e}_{j-1}^\top$, $s = 1, \dots, S$, where the empty sum is defined to be 0. One can see that $C_1 = 0$ and $C_S = A$. Since

$$(I - C_s)A^{(s)} = \left(I - \sum_{j=2}^s \alpha_j \mathbf{e}_j \mathbf{e}_{j-1}^\top \right) \left(I + \alpha_s \mathbf{e}_s \mathbf{e}_{s-1}^\top \right) = I - \sum_{j=2}^{s-1} \alpha_j \mathbf{e}_j \mathbf{e}_{j-1}^\top = I - C_{s-1}$$

for all $s = 2, \dots, S$, we have by iteration that

$$\begin{aligned} (I - A)A^{(S)} \dots A^{(2)} &= (I - C_S)A^{(S)} \dots A^{(2)} = (I - C_{S-1})A^{(S-1)} \dots A^{(2)} \\ &= \dots = (I - C_2)A^{(2)} = I - C_1 = I \quad \square \end{aligned}$$

Proof of Lemma 1. (i) \Leftrightarrow (ii) Since $\det(I - A) = 1$, by Laplace's formula, we have

$$\det(zI - (I - A)^{-1}B) = \det^{-1}(I - A) \det(z(I - A) - B) = Q(z)$$

i.e., Q is the characteristic polynomial of the matrix $(I - A)^{-1}B$ and the assertion follows.

(iii) \Leftrightarrow (iv) By Laplace's formula P is the characteristic polynomial of the matrix $A + B$ and the statement follows.

(ii) \Leftrightarrow (iii) By (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (iv) Q and P are the characteristic polynomials of non-negative matrices. Thus, by the Perron theorem (see Theorem 8.3.1 in Horn and Johnson (2012)), the maximal roots in modulus of the polynomials Q and P are non-negative real numbers. Thus, it is enough to show that (a) $Q(z) > 0$ for all $z \geq 1$ if and only if (b) $P(z) > 0$ for all $z \geq 1$. On the one hand, $Q(z) \leq P(z)$ if $z \geq 1$ implies (a) \Rightarrow (b). On the other hand, since

$$\begin{aligned} 0 < P(1) &= \prod_{j=1}^S (1 - \beta_j) - \prod_{j=1}^S \alpha_j \leq \prod_{j=1}^S (1 - \beta_j z^{-1}) - \prod_{j=1}^S \alpha_j \\ &= z^{-S} \left(\prod_{j=1}^S (z - \beta_j) - z^S \prod_{j=1}^S \alpha_j \right) \leq z^{-S} Q(z), \end{aligned}$$

for all $z \geq 1$, we have (b) \Rightarrow (a). \square

Proof of Lemma 2. By definition of A and B , $M := A + B$ is a strictly substochastic matrix since $\sum_{j=1}^S m_{i,j} = \alpha_i + \beta_i < 1$ for all $i = 1, \dots, S$. By Theorem 8.1.22 in Horn and Johnson (2012), we obtain $\rho(A) \leq \max_{1 \leq i \leq S} \sum_{j=1}^S m_{i,j} < 1$. \square

Proof of Lemma 3. (i) The recursion for Y_s , $s = 1, \dots, S$, follows from the form of A immediately. Define the \mathbb{N}_0^S -valued r.v.'s $\mathbf{Z}_s := (Y_1, \dots, Y_s, Z_{s+1}, \dots, Z_S)^\top$, $s = 1, \dots, S - 1$,

and let $\mathbf{Z}_S := \mathbf{Y}$. Since $Y_1 = Z_1$ by the form of A , we have $\mathbf{Z}_1 = \mathbf{Z}$. By the recursion, one can see that $\mathbf{Z}_s = A^{(s)} \circ \mathbf{Z}_{s-1}$ for all $s = 2, \dots, S$ which implies (12).

(ii) By recursion, (12) implies that \mathbf{Y} has finite mean provided \mathbf{Z} has a finite first moment. By the law of total conditional expectation and (4), we have

$$\mathbf{E}(A \circ \mathbf{Y} | \mathcal{F}) = \mathbf{E}(\mathbf{E}(A \circ \mathbf{Y} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) = \mathbf{E}(A\mathbf{Y} | \mathcal{F}) = A\mathbf{E}(\mathbf{Y} | \mathcal{F}).$$

Thus, by taking conditional expectation of (11), we have

$$\mathbf{E}(\mathbf{Y} | \mathcal{F}) = A\mathbf{E}(\mathbf{Y} | \mathcal{F}) + \mathbf{E}(\mathbf{Z} | \mathcal{F})$$

which implies (13). The formulas for $\mathbf{E}(\mathbf{Y} | \mathbf{Z})$ and $\mathbf{E}(\mathbf{Y})$ follow immediately.

(iii) By recursion, (12) implies that \mathbf{Y} has finite variance matrix provided \mathbf{Z} has finite second moment. Clearly, by (11), we have $\mathbf{Y} = (I - A)^{-1}(A \circ \mathbf{Y} - A\mathbf{Y} + \mathbf{Z})$. By the law of total conditional covariance, we have

$$\begin{aligned} \text{Cov}(A \circ \mathbf{Y} - A\mathbf{Y}, \mathbf{Z} | \mathcal{F}) &= \text{Cov}(\mathbf{E}(A \circ \mathbf{Y} - A\mathbf{Y} | \mathbf{Y}, \mathcal{F}), \mathbf{E}(\mathbf{Z} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) \\ &\quad + \mathbf{E}(\text{Cov}(A \circ \mathbf{Y} - A\mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) = 0 \end{aligned}$$

because the first term is 0 since (4) implies $\mathbf{E}(A \circ \mathbf{Y} | \mathbf{Y}, \mathcal{F}) = A\mathbf{Y}$, and the second term is 0 since

$$\begin{aligned} \text{Cov}(A \circ \mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) &= \text{Cov}(\mathbf{E}(A \circ \mathbf{Y} | \mathbf{Y}, \mathcal{F}), \mathbf{E}(\mathbf{Z} | \mathbf{Y}, \mathcal{F}) | \mathbf{Y}, \mathcal{F}) \\ &\quad + \mathbf{E}(\text{Cov}(A \circ \mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) | \mathbf{Y}, \mathcal{F}) \\ &= \text{Cov}(A\mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) = A \text{Cov}(\mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) \end{aligned}$$

again by the law of total conditional covariance, (4), and $\text{Cov}(A \circ \mathbf{Y}, \mathbf{Z} | \mathbf{Y}, \mathcal{F}) = 0$. Thus, by (13), we have

$$\begin{aligned} \text{Var}(\mathbf{Y} | \mathcal{F}) &= \text{Var}((I - A)^{-1}(A \circ \mathbf{Y} - A\mathbf{Y} + \mathbf{Z} | \mathcal{F})) \\ &= (I - A)^{-1} (\text{Var}(A \circ \mathbf{Y} - A\mathbf{Y} | \mathcal{F}) + \text{Var}(\mathbf{Z} | \mathcal{F})) ((I - A)^{-1})^\top \end{aligned}$$

Hence, (14) is derived since, by the law of total conditional variance, we obtain

$$\begin{aligned} \text{Var}(A \circ \mathbf{Y} - A\mathbf{Y} | \mathcal{F}) &= \text{Var}(\mathbf{E}(A \circ \mathbf{Y} - A\mathbf{Y} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) + \mathbf{E}(\text{Var}(A \circ \mathbf{Y} - A\mathbf{Y} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) \\ &= \mathbf{E}(\text{Var}(A \circ \mathbf{Y} | \mathbf{Y}, \mathcal{F}) | \mathcal{F}) = \mathbf{E}(\text{diag}(V_A \mathbf{Y}) | \mathcal{F}) = \text{diag}(V_A \mathbf{E}(\mathbf{Y} | \mathcal{F})) \end{aligned}$$

where the first part is 0 by (4), and in the second part the conditional variance formula of (4) is applied, and finally $\mathbf{E}(\mathbf{Y} | \mathcal{F})$ is replaced by (13). The last assertion for $\text{Var}(\mathbf{Y} | \mathbf{Z})$ follows by $\text{Var}(\mathbf{Z} | \mathbf{Z}) = 0$, while the formula for $\text{Var}(\mathbf{Y})$ can be derived by taking expectation of (14) and using the law of total conditional variance. \square

Proof of covariance formula in Remark 2. By the law of total conditional covariance, we have

$$\begin{aligned}\text{Cov}(\mathbf{Y}, \mathbf{Z}|\mathcal{F}) &= \text{Cov}(\mathbb{E}(\mathbf{Y}|\mathbf{Z}, \mathcal{F}), \mathbb{E}(\mathbf{Z}|\mathbf{Z}, \mathcal{F})|\mathcal{F}) + \mathbb{E}(\text{Cov}(\mathbf{Y}, \mathbf{Z}|\mathbf{Z}, \mathcal{F})|\mathcal{F}) \\ &= \text{Cov}((I - A)^{-1}\mathbf{Z}, \mathbf{Z}|\mathcal{F}) = (I - A)^{-1} \text{Var}(\mathbf{Z}|\mathcal{F})\end{aligned}$$

since $\text{Cov}(\mathbf{Y}, \mathbf{Z}|\mathbf{Z}, \mathcal{F}) = 0$. □

Proof of Lemma 4. For all $n, n' \in \mathbb{N}_0$ and binomial thinning operator $\alpha \circ$ with counting sequence $\{\xi_j\}$ we have

$$\mathbb{E}|\alpha \circ n - \alpha \circ n'| = \alpha(n \vee n' - n \wedge n') = \alpha|n - n'|$$

since $|\alpha \circ n - \alpha \circ n'| = \sum_{j=n \wedge n'+1}^{n \vee n'} \xi_j$. Thus, for all $\mathbf{n}, \mathbf{n}' \in \mathbb{N}_0^S$, we have $\mathbb{E}|M \circ \mathbf{n} - M \circ \mathbf{n}'|_{\text{vec}} \leq M|\mathbf{n} - \mathbf{n}'|_{\text{vec}}$ since

$$\mathbb{E} \left| \sum_{j=1}^S m_{i,j} \circ n_j - \sum_{j=1}^S m_{i,j} \circ n'_j \right| \leq \sum_{j=1}^S \mathbb{E}|m_{i,j} \circ n_j - m_{i,j} \circ n'_j| = \sum_{j=1}^S m_{i,j}|n_j - n'_j|$$

By the law of total conditional expectation, we have the statement of the lemma as

$$\begin{aligned}\mathbb{E}|M \circ \mathbf{Z} - M \circ \mathbf{Z}'|_{\text{vec}} &= \mathbb{E}(\mathbb{E}(|M \circ \mathbf{Z} - M \circ \mathbf{Z}'|_{\text{vec}}|\mathbf{Z}, \mathbf{Z}')) \\ &\leq \mathbb{E}(M|\mathbf{Z} - \mathbf{Z}'|_{\text{vec}}) = M\mathbb{E}|\mathbf{Z} - \mathbf{Z}'|_{\text{vec}} \quad \square\end{aligned}$$

Proof of Equation (17). We prove the statement by induction. The statement is true for $n = 1$ since $\mathbf{Y}_k^{(0)} = 0$ and $\mathbf{Y}_k^{(1)} = \mathbf{Z}_k^{(1)}$ for all $k \in \mathbb{Z}$. Assume that (17) holds for $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\mathbf{Y}_k^{(n+1)} &= \mathbf{Y}_k^{(n)} + \mathbf{Z}_k^{(n+1)} \\ &= (I - A_k)^{\circ-1} \left(B_k \circ \mathbf{Y}_{k-1}^{(n-1)} + \varepsilon_k \right) + (I - A_k)^{\circ-1}_{|B_k \circ \mathbf{Y}_{k-1}^{(n-1)} + \varepsilon_k} \left(B_k \circ_{|\mathbf{Y}_{k-1}^{(n-1)}} \mathbf{Z}_{k-1}^{(n)} \right) \\ &= (I - A_k)^{\circ-1} \left(B_k \circ \mathbf{Y}_{k-1}^{(n-1)} + \varepsilon_k + B_k \circ_{|\mathbf{Y}_{k-1}^{(n-1)}} \mathbf{Z}_{k-1}^{(n)} \right) \\ &= (I - A_k)^{\circ-1} \left(B_k \circ (\mathbf{Y}_{k-1}^{(n-1)} + \mathbf{Z}_{k-1}^{(n)}) + \varepsilon_k \right) \\ &= (I - A_k)^{\circ-1} \left(B_k \circ \mathbf{Y}_{k-1}^{(n)} + \varepsilon_k \right) \quad \square\end{aligned}$$

Proof of the properties of $\{\mathbf{Y}_k^{(n)}\}$ and $\{\mathbf{Z}_k^{(n)}\}$. For each $k \in \mathbb{Z}$, define the probability space $(\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$ as $\Omega_k := (\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}})^S \times \mathbb{N}^S$, the σ -algebra \mathcal{A}_k generated by the cylinder sets of Ω_k , and, for each $\omega_k = ((\mathbf{x}_s, \mathbf{y}_s)_{s=1, \dots, S}, \mathbf{n})$ where $\mathbf{x}_s \in \{0, 1\}^{N_s}$, $\mathbf{y}_s \in \{0, 1\}^{M_s}$, $N_s, M_s \in \mathbb{N}$, $s = 1, \dots, S$, and $\mathbf{n} \in \mathbb{N}_0^S$, $\mathbb{P}_k(\omega_k) := p_{\mathbf{n}} \prod_{s=1}^S \prod_{i=1}^{N_s} \alpha_s^{x_{s,i}} (1 - \alpha_s)^{1-x_{s,i}} \prod_{j=1}^{M_s} \beta_s^{y_{s,j}} (1 - \beta_s)^{1-y_{s,j}}$, where $p_{\mathbf{n}} := \mathbb{P}(\varepsilon_k = \mathbf{n})$, $\mathbf{n} \in \mathbb{N}_0^S$. Then, ω_k represents a cylinder set in Ω_k whose set generates all cylinder sets. Thus, \mathbb{P}_k can be uniquely extended to the σ -algebra \mathcal{A}_k as a

probability measure by Theorem 8.2.2 in Dudley (2004). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the infinite product of $(\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$, $k \in \mathbb{Z}$, which is well defined by Theorem 8.2.2 in Dudley (2004) too. By construction, the canonical process $\{U_k\}$, defined by $U_k(\omega) := \omega_k$ for all $\omega = (\omega_k) \in \Omega$, follows the distribution of independent r.v.'s $\{\xi_{k,s,j}\}$, $\{\eta_{k,s,j}\}$, $\{\varepsilon_k\}$. We prove the \mathcal{G}_k -measurability of $\mathbf{Y}_k^{(n)}$ and $\mathbf{Z}_k^{(n+1)}$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ by induction. Clearly, $\mathbf{Y}_k^{(0)}$ is \mathcal{G}_k -measurable for all $k \in \mathbb{Z}$. Suppose that, for a fixed $n \in \mathbb{N}_0$, the r.v. $\mathbf{Y}_k^{(n)}$ is \mathcal{G}_k -measurable for all $k \in \mathbb{Z}$. Then, by (19), $\mathbf{Y}_k^{(n+1)}$ depends on $\{\xi_{k,s,j}\}_{s,j}$, $\{\eta_{k,s,j}\}_{s,j}$, $\{\varepsilon_{kS+s}\}_s$, and $\mathbf{Y}_{k-1}^{(n)}$, hence $\mathbf{Y}_k^{(n+1)}$ is \mathcal{G}_k -measurable as well. Similarly, $\mathbf{Z}_k^{(1)}$ is \mathcal{G}_k -measurable by definition, and the induction follows by (16).

For all $h \in \mathbb{Z}$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, define the probability distribution $\mathbb{P}_{m,h}^{(n)}$ on \mathbb{N}_0^{mS} as

$$\mathbb{P}_{m,h}^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_m) := \mathbb{P}\left(\mathbf{Y}_{h+1}^{(n)} = \mathbf{y}_1, \dots, \mathbf{Y}_{h+m}^{(n)} = \mathbf{y}_m\right)$$

where $\mathbf{y}_1, \dots, \mathbf{y}_m \in \mathbb{N}_0^S$. To prove that $\{\mathbf{Y}_k^{(n)}\}_k$ is strictly stationary, it is sufficient to show that $\mathbb{P}_{m,h}^{(n)}$ does not depend on h . This statement is true for $n = 0$. Suppose that $\mathbb{P}_{m,h}^{(n)}$ does not depend on h for a $n \in \mathbb{N}_0$. By the law of total probability, the recursion (17) and since the sequence $\{(A_k \circ, B_k \circ, \varepsilon_k)\}$ is i.i.d., we obtain, for all $h \in \mathbb{Z}$ and $m \in \mathbb{N}$, that

$$\mathbb{P}_{m,h}^{(n+1)}(\mathbf{y}_1, \dots, \mathbf{y}_m) = \sum_{\mathbf{z}_1, \dots, \mathbf{z}_m \in \mathbb{N}_0^S} \mathbb{P}_{m,h-1}^{(n)}(\mathbf{z}_1, \dots, \mathbf{z}_m) \prod_{j=1}^m \mathbb{P}\left((I - A)^{\circ-1}(B \circ \mathbf{z}_j + \varepsilon) = \mathbf{y}_j\right)$$

for all $\mathbf{y}_1, \dots, \mathbf{y}_m \in \mathbb{N}_0^S$. In this formula, the right-hand side does not depend on h . Thus, by induction, $\{\mathbf{Y}_k^{(n)}\}_k$ is strictly stationary for all $n \in \mathbb{N}_0$. To prove the non-anticipativity, we note that the \mathbb{N}_0^S -valued r.v.'s $\mathbf{Y}_j^{(n)}$, $j < k$, are \mathcal{G}_{k-1} -measurable for all $n \in \mathbb{N}_0$. Since \mathcal{G}_{k-1} and $\{\xi_{l,s,j}, \eta_{l,s,j}, \varepsilon_{lS+s} \mid j \in \mathbb{N}, s = 1, \dots, S, l \geq k\}$ are mutually independent for all $k \in \mathbb{Z}$, the non-anticipativity follows. Finally, to prove the ergodicity, let $T : \Omega \rightarrow \Omega$ denote the bilateral shift operator on $(\Omega, \mathcal{A}, \mathbb{P})$, see page 271 in Dudley (2004). By Theorem 8.4.5 in Dudley (2004) T is an ergodic measure preserving transformation. By construction, if $T(\xi)(\omega) := \xi(T\omega)$ for all $\omega \in \Omega$ and r.v. ξ , then one can see that $T(\xi_{k,s,j}) = \xi_{k+1,s,j}$, $T(\eta_{k,s,j}) = \eta_{k+1,s,j}$, and $T(\varepsilon_{kS+s}) = \varepsilon_{(k+1)S+s}$ for all $k \in \mathbb{Z}$, $j \in \mathbb{N}$ and $s = 1, \dots, S$. We prove by induction that $T(\mathbf{Y}_k^{(n)}) = \mathbf{Y}_{k+1}^{(n)}$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. This statement is true for $n = 0$. Suppose that the statement holds for a $n \in \mathbb{N}_0$. Then, by (19), we have the statement for $n + 1$. Thus, $\mathbf{Y}_k^{(n)}(T\omega) = \mathbf{Y}_{k+1}^{(n)}(\omega)$ for all $\omega \in \Omega$, $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ which implies that $\{\mathbf{Y}_k^{(n)}\}_k$ is ergodic for all $n \in \mathbb{N}_0$. By construction and assumptions of the PINAR(1, 1_S) model, $\{\mathbf{Z}_k^{(1)}\}_k$ is a sequence of independent \mathbb{N}_0^S -valued r.v.'s. Moreover, the r.v.'s involved from the counting sequences into the definition of $\mathbf{Z}_k^{(n)}$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, are mutually independent by (16). Hence, for each $k \in \mathbb{Z}$, the r.v. $\mathbf{Z}_k^{(n)}$ depends on $\mathbf{Z}_{k-n+1}^{(1)}$ and a set \mathcal{R}_n of r.v.'s for all $n \in \mathbb{N}$, which are mutually independent. This implies that the sequence $\{\mathbf{Z}_k^{(n)}\}_n$ consists of independent r.v.' for all $k \in \mathbb{Z}$.

One can easily see that the \mathbb{N}_0 -valued r.v.'s $\mathbf{Y}_k^{(n)}$ and $\mathbf{Z}_k^{(n+1)}$ are measurable w.r.t. the σ -algebra generated by the r.v.'s $\{\xi_{l,s,j}, \eta_{l,s,j}, \varepsilon_{lS+s} \mid j \in \mathbb{N}, s = 1, \dots, S, k-n \leq l \leq k\}$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, $\{\mathbf{Y}_k^{(n)}\}_k$ is an n -dependent process, see Definition 6.4.3 in Brockwell and Davis (2013). This fact also implies the ergodicity of $\{\mathbf{Y}_k^{(n)}\}_k$ for all $n \in \mathbb{N}$.

We apply recursion (18) and Lemma 3 by choosing $\mathbf{Y} = \mathbf{Y}_k^{(n)}$, $\mathbf{Z} = B_k \circ \mathbf{Y}_{k-1}^{(n)} + \varepsilon_k$ and $\mathcal{F} = \mathcal{G}_{k-1}$. By (4), we have $\mathbb{E}(B_k \circ \mathbf{Y}_{k-1}^{(n)} + \varepsilon_k | \mathcal{G}_{k-1}) = B\mathbf{Y}_{k-1}^{(n)} + \boldsymbol{\lambda}$ and $\text{Var}(B_k \circ \mathbf{Y}_{k-1}^{(n)} + \varepsilon_k | \mathcal{G}_{k-1}) = \text{diag}(V_B \mathbf{Y}_{k-1}^{(n)}) + \Sigma_\varepsilon$. Using (18) and Lemma 3, we have

$$\mathbb{E}\left(\mathbf{Y}_k^{(n)} | \mathcal{G}_{k-1}\right) = (I - A)^{-1}(B\mathbf{Y}_{k-1}^{(n-1)} + \boldsymbol{\lambda}), \quad (\text{A1})$$

$$\begin{aligned} \text{Var}\left(\mathbf{Y}_k^{(n)} | \mathcal{G}_{k-1}\right) &= (I - A)^{-1} \left(\text{diag}(V_A(I - A)^{-1}(B\mathbf{Y}_{k-1}^{(n-1)} + \boldsymbol{\lambda}) + V_B \mathbf{Y}_{k-1}^{(n-1)}) \right. \\ &\quad \left. + \Sigma_\varepsilon \right) ((I - A)^{-1})^\top \end{aligned} \quad (\text{A2})$$

Clearly, $\boldsymbol{\mu}^{(0)} = \mathbb{E}(\mathbf{Y}_k^{(0)}) = 0$ for all $k \in \mathbb{Z}$. By taking expectation of recursion (A1), for any $n \in \mathbb{N}$, we have

$$\mathbb{E}\left(\mathbf{Y}_k^{(n)}\right) = (I - A)^{-1} B \mathbb{E}\left(\mathbf{Y}_{k-1}^{(n-1)}\right) + (I - A)^{-1} \boldsymbol{\lambda} = \sum_{j=0}^{n-1} ((I - A)^{-1} B)^j (I - A)^{-1} \boldsymbol{\lambda}$$

for all $k \in \mathbb{Z}$, where the right-hand side is finite and does not depend on k . Moreover, this implies that $\{\boldsymbol{\mu}^{(n)}\}$ satisfies the recursion (20). Under Assumption 1, by Lemma 1 and Corollary 5.6.16 in Horn and Johnson (2012), the infinite series $\sum_{j=0}^{\infty} ((I - A)^{-1} B)^j$ is convergent and the limit is $(I - (I - A)^{-1} B)^{-1}$. Thus, we obtain by (26) that

$$0 \leq \boldsymbol{\mu}^{(n)} \leq \sum_{j=0}^{\infty} ((I - A)^{-1} B)^j (I - A)^{-1} \boldsymbol{\lambda} = (I - (I - A)^{-1} B)^{-1} (I - A)^{-1} \boldsymbol{\lambda} = \boldsymbol{\mu}$$

for all $n \in \mathbb{N}_0$. Hence $\{\boldsymbol{\mu}^{(n)}\}$ is non-decreasing, bounded and the limit of $\{\boldsymbol{\mu}^{(n)}\}$ is $\boldsymbol{\mu}$. The recursion for $\{\boldsymbol{\nu}^{(n)}\}$ follows from (16) by Lemma 3.

Clearly, $\Sigma_0 = \text{Var}(\mathbf{Y}_k^{(0)}) = O_{S \times S}$ for all $k \in \mathbb{Z}$. By the law of total conditional variance, (A1), (A2), for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \text{Var}\left(\mathbf{Y}_k^{(n)}\right) &= \text{Var}\left(\mathbb{E}(\mathbf{Y}_k^{(n)} | \mathcal{G}_{k-1})\right) + \mathbb{E}\left(\text{Var}(\mathbf{Y}_k^{(n)} | \mathcal{G}_{k-1})\right) \\ &= (I - A)^{-1} \left(B \text{Var}(\mathbf{Y}_{k-1}^{(n-1)}) B^\top + \text{diag}(V_A \boldsymbol{\mu}^{(n)} + V_B \boldsymbol{\mu}^{(n-1)}) + \Sigma_\varepsilon \right) ((I - A)^{-1})^\top \\ &= \sum_{j=0}^{n-1} ((I - A)^{-1} B)^j (I - A)^{-1} \left(\text{diag}(V_A \boldsymbol{\mu}^{(n-j)} + V_B \boldsymbol{\mu}^{(n-j-1)}) \right. \\ &\quad \left. + \Sigma_\varepsilon \right) ((I - A)^{-1})^\top \left(((I - A)^{-1} B)^j \right)^\top \end{aligned}$$

for all $k \in \mathbb{Z}$, where the right-hand side does not depend on k . Thus, $\Sigma^{(n)}$ is finite for all $n \in \mathbb{N}$, and the sequence $\{\Sigma^{(n)}\}$ satisfies the recursion (21). Clearly, $\Sigma^{(1)} - \Sigma^{(0)} = \Sigma^{(1)}$ is a

symmetric positive semi-definite matrix. By (21), we have

$$\Sigma^{(n+1)} - \Sigma^{(n)} = (I - A)^{-1} \left(B \left(\Sigma^{(n)} - \Sigma^{(n-1)} \right) B^\top + \text{diag}(V_A \Delta_{n+1} + V_B \Delta_n) \right) \left((I - A)^{-1} \right)^\top$$

where $\Delta_n := \boldsymbol{\mu}^{(n)} - \boldsymbol{\mu}^{(n-1)} \geq 0$ for all $n \in \mathbb{N}$. Thus, by induction, $\{\Sigma^{(n)}\}$ is monotone non-decreasing. By using the vec operator, we have the following expression for the vectorized $\Sigma^{(n)}$

$$\text{vec} \Sigma^{(n)} = \sum_{j=0}^{n-1} \left(\left((I - A)^{-1} B \right)^{\otimes 2} \right)^j \left((I - A)^{-1} \right)^{\otimes 2} \text{vec} \left(\text{diag}(V_A \boldsymbol{\mu}^{(n-j)} + V_B \boldsymbol{\mu}^{(n-j-1)}) + \Sigma_\epsilon \right)$$

Under Assumption 1, see the proof of Proposition 1, $\rho(\left((I - A)^{-1} B \right)^{\otimes 2}) < 1$. Thus, Corollary 5.6.16 in Horn and Johnson (2012), implies that the infinite series $\sum_{j=0}^{\infty} \left(\left((I - A)^{-1} B \right)^{\otimes 2} \right)^j$ is convergent. Since $\boldsymbol{\mu}^{(n)} \rightarrow \boldsymbol{\mu}$ as $n \rightarrow \infty$, one can see that $\text{vec} \Sigma^{(n)}$ converges to the right hand side of (27) and thus the sequence $\{\Sigma^{(n)}\}$ is bounded and $\Sigma^{(n)} \rightarrow \Sigma$ as $n \rightarrow \infty$. The proof of the recursion for $\{\Phi^{(n)}\}$ is similar.

Clearly, $\Gamma^{(0)}(h) = O_{S \times S}$ for all $h \in \mathbb{Z}$. By the law of total covariance and (A1), we have, for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and $m, h \in \mathbb{N}$,

$$\begin{aligned} \text{Cov}(\mathbf{Y}_{k+h}^{(m)}, \mathbf{Y}_k^{(n)}) &= \text{Cov} \left(\mathbb{E}(\mathbf{Y}_{k+h}^{(m)} | \mathcal{G}_{k+h-1}), \mathbb{E}(\mathbf{Y}_k^{(n)} | \mathcal{G}_{k+h-1}) \right) + \mathbb{E} \left(\text{Cov}(\mathbf{Y}_{k+h}^{(m)}, \mathbf{Y}_k^{(n)} | \mathcal{G}_{k+h-1}) \right) \\ &= \text{Cov} \left((I - A)^{-1} B \mathbf{Y}_{k+h-1}^{(m-1)} + (I - A)^{-1} \boldsymbol{\lambda}, \mathbf{Y}_k^{(n)} \right) \\ &= (I - A)^{-1} B \text{Cov}(\mathbf{Y}_{k+h-1}^{(m-1)}, \mathbf{Y}_k^{(n)}) \end{aligned}$$

since the second term in the total covariance formula is 0 by the \mathcal{G}_{k+h-1} -measurability of $\mathbf{Y}_k^{(n)}$. Thus, we obtain by iterating that, for all $k \in \mathbb{Z}$ and $n, h \in \mathbb{N}$,

$$\Gamma^{(n)}(h) = \text{Cov} \left(\mathbf{Y}_{k+h}^{(n)}, \mathbf{Y}_k^{(n)} \right) = \begin{cases} \left((I - A)^{-1} B \right)^h \text{Cov} \left(\mathbf{Y}_k^{(n-h)}, \mathbf{Y}_k^{(n)} \right) & \text{if } 0 \leq h \leq n, \\ \left((I - A)^{-1} B \right)^n \text{Cov} \left(\mathbf{Y}_{k+h-n}^{(0)}, \mathbf{Y}_k^{(n)} \right) & \text{if } n < h. \end{cases}$$

On the one hand, this implies that $\Gamma^{(n)}(h) = O_{S \times S}$ if $n < h$. On the other hand, we prove that $\text{Cov}(\mathbf{Y}_k^{(n-h)}, \mathbf{Y}_k^{(n)}) = \Sigma^{(n-h)}$ for all $0 \leq h \leq n \in \mathbb{N}$. Let $\mathbf{U}_k^{(n)} := B_k \circ \mathbf{Y}_{k-1}^{(n)} + \boldsymbol{\varepsilon}_k$ and $\mathbf{V}_k^{(n)} := B_k \circ_{|\mathbf{Y}_{k-1}^{(n-1)}} \mathbf{Z}_{k-1}^{(n)}$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Moreover, define the σ -algebra $\mathcal{H}_k \subset \mathcal{A}$ generated by the \mathbb{N}_0 -valued r.v.'s $\{\mathbf{U}_l^{(n)} \mid l \leq k, n \in \mathbb{N}\}$ for all $k \in \mathbb{Z}$. Then, by (i), $A_k \circ$ and \mathcal{H}_k are mutually independent. We have, by (15), (16), Remark 1, and the law of total

covariance, for all $k \in \mathbb{Z}$, $n, m \in \mathbb{N}$ that

$$\begin{aligned}
\text{Cov}\left(\mathbf{Y}_k^{(n)}, \mathbf{Z}_k^{(n+m)}\right) &= \text{Cov}\left((I - A_k)^{\circ-1} \mathbf{U}_k^{(n-1)}, (I - A_k)_{|\mathbf{U}_k^{(n+m-2)}}^{\circ-1} \mathbf{V}_k^{(n+m-1)}\right) \\
&= \text{Cov}\left(\mathbb{E}\left((I - A_k)^{\circ-1} \mathbf{U}_k^{(n-1)} | \mathcal{H}_k\right), \mathbb{E}\left((I - A_k)_{|\mathbf{U}_k^{(n+m-2)}}^{\circ-1} \mathbf{V}_k^{(n+m-1)} | \mathcal{H}_k\right)\right) \\
&\quad + \mathbb{E}\left(\text{Cov}\left((I - A_k)^{\circ-1} \mathbf{U}_k^{(n-1)}, (I - A_k)_{|\mathbf{U}_k^{(n+m-2)}}^{\circ-1} \mathbf{V}_k^{(n+m-1)} | \mathcal{H}_k\right)\right) \\
&= (I - A)^{-1} \text{Cov}\left(\mathbf{U}_k^{(n-1)}, \mathbb{E}(\mathbf{V}_k^{(n+m-1)} | \mathcal{H}_k)\right) ((I - A)^{-1})^\top \\
&= (I - A)^{-1} \text{Cov}\left(\mathbf{U}_k^{(n-1)}, \mathbf{V}_k^{(n+m-1)}\right) ((I - A)^{-1})^\top
\end{aligned}$$

since the second term is 0 in the total covariance formula because the monotonicity of $\{\mathbf{Y}_k^{(n)}\}_n$ implies $\mathbf{U}_k^{(n-1)} \leq \mathbf{U}_k^{(n+m-2)}$ and $\text{Cov}(\mathbf{Y}, \mathbb{E}(\mathbf{Z} | \mathcal{F})) = \text{Cov}(\mathbf{Y}, \mathbf{Z})$ if \mathbf{Y} is \mathcal{F} -measurable. Moreover, for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, we have

$$\begin{aligned}
\text{Cov}\left(\mathbf{U}_k^{(n)}, \mathbf{V}_k^{(n+m)}\right) &= \text{Cov}\left(\mathbb{E}(\mathbf{U}_k^{(n)} | \mathcal{G}_{k-1}), \mathbb{E}(\mathbf{V}_k^{(n+m)} | \mathcal{G}_{k-1})\right) + \mathbb{E}\left(\text{Cov}(\mathbf{U}_k^{(n)}, \mathbf{V}_k^{(n+m)} | \mathcal{G}_{k-1})\right) \\
&= \text{Cov}\left(B\mathbf{Y}_{k-1}^{(n)} + \boldsymbol{\lambda}, B\mathbf{Z}_{k-1}^{(n+m)}\right) = B \text{Cov}\left(\mathbf{Y}_{k-1}^{(n)}, \mathbf{Z}_{k-1}^{(n+m)}\right) B^\top
\end{aligned}$$

since

$$\text{Cov}\left(\mathbf{U}_k^{(n)}, \mathbf{V}_k^{(n+m)} | \mathcal{G}_{k-1}\right) = \text{Cov}\left(B_k \circ \mathbf{Y}_{k-1}^{(n)} + \boldsymbol{\varepsilon}_k, B_k \circ_{|\mathbf{Y}_{k-1}^{(n+m-1)}} \mathbf{Z}_{k-1}^{(n+m)} | \mathcal{G}_{k-1}\right) = 0$$

because $\mathbf{Y}_{k-1}^{(n)} \leq \mathbf{Y}_{k-1}^{(n+m-1)}$ and $\boldsymbol{\varepsilon}_k$ is independent of \mathcal{G}_{k-1} . Thus, we have

$$\begin{aligned}
\text{Cov}\left(\mathbf{Y}_k^{(n)}, \mathbf{Z}_k^{(n+m)}\right) &= (I - A)^{-1} B \text{Cov}\left(\mathbf{Y}_{k-1}^{(n-1)}, \mathbf{Z}_{k-1}^{(n+m-1)}\right) ((I - A)^{-1} B)^\top \\
&= ((I - A)^{-1} B)^n \text{Cov}\left(\mathbf{Y}_{k-n}^{(0)}, \mathbf{Z}_{k-n}^{(m)}\right) (((I - A)^{-1} B)^n)^\top = 0
\end{aligned}$$

This implies that

$$\text{Cov}\left(\mathbf{Y}_k^{(n-h)}, \mathbf{Y}_k^{(n)}\right) = \text{Var}\left(\mathbf{Y}_k^{(n-h)}\right) + \sum_{j=n-h+1}^n \text{Cov}\left(\mathbf{Y}_k^{(n-h)}, \mathbf{Z}_k^{(j)}\right) = \Sigma^{(n-h)}$$

which completes the proof. \square

Proof of Proposition 1. Since for a stationary solution $\boldsymbol{\mu}_k = \boldsymbol{\mu}_{k-1} = \boldsymbol{\mu}$ we have (22) by (37), (note that (22) is the same as the equation at the bottom of p. 547 in Filho et al. (2021)). Since $\Sigma_k = \Sigma_{k-1} = \Sigma$, we deduce (23) from (40). Under Assumption 1, $\rho((I - A)^{-1} B) < 1$ by Lemma 1, hence the first part of (24) is an immediate consequence of (22), while the second part follows from

$$(I - (I - A)^{-1} B)^{-1} (I - A)^{-1} (I - A - B) = (I - (I - A)^{-1} B)^{-1} (I - (I - A)^{-1} B) = I$$

Finally, $\rho((I_S - A)^{-1} B) < 1$ implies $\rho(((I_S - A)^{-1} B)^{\otimes 2}) < 1$, thus the matrix $I_{S^2} - ((I_S - A)^{-1} B)^{\otimes 2}$ is also non-singular. Applying (41) in case of $\Sigma_k = \Sigma_{k-1} = \Sigma$ and solving the linear equation, we obtain (25). \square

Proof of the properties of innovation sequence $\{\mathbf{M}_k\}$. Since $\Gamma(k, \ell) = \mathbf{E}(\mathbf{X}_k \mathbf{X}_\ell^\top)$ for all $k, \ell \in \mathbb{Z}$ and, by (32), $\mathbf{M}_k = (I - A)\mathbf{X}_k - B\mathbf{X}_{k-1}$ for all $k \in \mathbb{Z}$, we have for all $k, \ell \in \mathbb{Z}$ that

$$\begin{aligned} \mathbf{E}(\mathbf{M}_k \mathbf{M}_\ell^\top) &= (I - A)\Gamma(k, \ell)(I - A)^\top - (I - A)\Gamma(k, \ell - 1)B^\top \\ &\quad - B\Gamma(k - 1, \ell)(I - A)^\top + B\Gamma(k - 1, \ell - 1)B^\top \end{aligned}$$

Thus, by (23) and (42), we obtain for $k = \ell$ that

$$\mathbf{E}(\mathbf{M}_k \mathbf{M}_k^\top) = (I - A)\Sigma(I - A)^\top - B\Sigma B^\top = \Sigma_{\mathbf{M}}$$

and for $k \geq \ell$

$$\begin{aligned} \mathbf{E}(\mathbf{M}_k \mathbf{M}_\ell^\top) &= (I - A) \left((I - A)^{-1} B \right)^{k-\ell} \Sigma (I - A)^\top - (I - A) \left((I - A)^{-1} B \right)^{k-\ell+1} \Sigma B^\top \\ &\quad - B \left((I - A)^{-1} B \right)^{k-\ell-1} \Sigma (I - A)^\top + B \left((I - A)^{-1} B \right)^{k-\ell} \Sigma B^\top = O_{S \times S} \end{aligned}$$

These prove that $\{\mathbf{M}_k\}$ is an uncorrelated sequence with variance matrix $\Sigma_{\mathbf{M}}$. \square

Proof of Proposition 2. To show that the infinite series on the right-hand side of (33) converges almost surely and in the mean square for all $k \in \mathbb{Z}$, by Proposition 3.1.1 of Brockwell and Davis (2013) and Proposition C.9 of Lütkepohl (2005), it is enough to see that $\sup_{k \in \mathbb{Z}} \mathbf{E}(\mathbf{M}_k^\top \mathbf{M}_k) = \text{tr} \Sigma_{\mathbf{M}} < +\infty$ and $\sum_{j=0}^{\infty} \|((I - A)^{-1} B)^j\|$ converges which follows from Assumption 1 and Theorem 5.6.15 in Horn and Johnson (2012). By iterating (32) we have

$$\mathbf{X}_k = ((I - A)^{-1} B)^n \mathbf{X}_{k-n} + \sum_{j=0}^{n-1} ((I - A)^{-1} B)^j \mathbf{W}_{k-j}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Thus, to show that \mathbf{X}_k is the limit of the infinite series on the right-hand side of (33) almost surely and in the mean square, it is enough to see that $((I - A)^{-1} B)^n \mathbf{X}_{k-n}$ converges to 0 as $n \rightarrow \infty$ almost surely and in mean square. Markov's inequality and Theorem 5.6.12 of Horn and Johnson (2012) imply, for all $\epsilon > 0$,

$$\begin{aligned} \mathbf{P} \left(\|((I - A)^{-1} B)^n \mathbf{X}_{k-n}\| \geq \epsilon \right) &\leq \mathbf{P} \left(\|((I - A)^{-1} B)^n\| \|\mathbf{X}_{k-n}\| \geq \epsilon \right) \\ &\leq \epsilon^{-2} \text{tr}(\Sigma) \|((I - A)^{-1} B)^n\|^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, thus the almost sure convergence follows. On the other hand, the inequality $0 \leq \mathbf{E} \|((I - A)^{-1} B)^n \mathbf{X}_{k-n}\|^2 \leq \text{tr}(\Sigma) \|((I - A)^{-1} B)^n\|^2 \rightarrow 0$ as $n \rightarrow \infty$ implies the convergence in mean square. \square

Proof of Remark 4. On the one hand, we have by (35) that, for all $k \in \mathbb{Z}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{V}_k^{(n)} &= \sum_{n=2}^{\infty} \left(\mathbf{Z}_k^{(n)} - (I - A)^{-1} B \mathbf{Z}_{k-1}^{(n-1)} \right) + \mathbf{Z}_k^{(1)} - \boldsymbol{\nu}^{(1)} = \mathbf{Y}_k - (I - A)^{-1} B \mathbf{Y}_{k-1} - \boldsymbol{\nu}^{(1)} \\ &= \mathbf{Y}_k - (I - A)^{-1} (B \mathbf{Y}_{k-1} + \boldsymbol{\lambda}) = \mathbf{Y}_k - \mathbf{E}(\mathbf{Y}_k | \mathcal{F}_{kS}) = \mathbf{W}_k \end{aligned}$$

On the other hand, by recursion of $\{\boldsymbol{\nu}^{(n)}\}$, we have the recursion

$$\mathbf{Z}_k^{(j+1)} - \boldsymbol{\nu}^{(j+1)} = \mathbf{V}_k^{(j+1)} + (I - A)^{-1}B \left(\mathbf{Z}_{k-1}^{(j)} - \boldsymbol{\nu}^{(j)} \right),$$

for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, which implies the formula for $\mathbf{Z}_k^{(n)} - \boldsymbol{\nu}^{(n)}$. \square

Proof of Equation (42). It is enough to prove the recursion $\Gamma(k, \ell) = (I - A)^{-1}B\Gamma(k-1, \ell)$ for all $k > \ell$. We have, by the law of total covariance, (35) and the \mathcal{F}_{kS} -measurability of \mathbf{Y}_ℓ , that

$$\begin{aligned} \text{Cov}(\mathbf{Y}_k, \mathbf{Y}_\ell) &= \text{Cov}(\mathbf{E}(\mathbf{Y}_k | \mathcal{F}_{kS}), \mathbf{E}(\mathbf{Y}_\ell | \mathcal{F}_{kS})) + \mathbf{E}(\text{Cov}(\mathbf{Y}_k, \mathbf{Y}_\ell | \mathcal{F}_{kS})) \\ &= \text{Cov}((I - A)^{-1}(B\mathbf{Y}_{k-1} + \boldsymbol{\lambda}), \mathbf{Y}_\ell) = (I - A)^{-1}B \text{Cov}(\mathbf{Y}_{k-1}, \mathbf{Y}_\ell) \end{aligned}$$

since $\text{Cov}(\mathbf{Y}_k, \mathbf{Y}_\ell | \mathcal{F}_{kS}) = 0$. \square

Proof of Equation (43). Let $i, j \in \{1, \dots, S\}$. For all $h \in \mathbb{N}_0$, we have

$$(\Gamma(h))_{i,j} = (\Gamma(h, 0))_{i,j} = \text{Cov}(Y_{hS+i}, Y_j) = R(hS + i, j)$$

On the other hand, for all $h \in \mathbb{N}$, we have

$$(\Gamma(-h))_{i,j} = (\Gamma(h))_{j,i} = R(hS + j, i) = R(i, hS + j) = R(-hS + i, j) \quad \square$$

Proof of Equation (44). By the definition of functions γ_j , $j \in \mathbb{Z}$, we obtain

$$\gamma_j(hS + s) = R(hS + s + j, j) = R(j, hS + s + j) = R(-hS - s + (s + j), s + j) = \gamma_{j+s}(-hS - s)$$

for all $h, j \in \mathbb{Z}$ and $s = 1, \dots, S$. \square

Proof of Yule-Walker equations. By the Yule-Walker (YW) equations of the weakly stationary process $\{\mathbf{Y}_k\}$, we have

$$(\Gamma(h))_{i,j} = \alpha_i(\Gamma(h - \delta_{1i}))_{i-1+\delta_{1i}S,j} + \beta_i(\Gamma(h-1))_{i,j}$$

for all $h \in \mathbb{N}$, $i, j = 1, \dots, S$ which implies (45). Let $h = 1$. On the one hand, applying (45) in cases of $i = j = s$ and $i = j + 1 = s$ for all $s = 2, \dots, S$ we have the following linear equation for the parameter vector $(\boldsymbol{\alpha}_s, \boldsymbol{\beta}_s)^\top$

$$\begin{bmatrix} \gamma_{s-1}(S) & \gamma_{s-1}(1) \\ \gamma_s(S-1) & \gamma_s(0) \end{bmatrix} \begin{bmatrix} \alpha_s \\ \beta_s \end{bmatrix} = \begin{bmatrix} \gamma_{s-1}(S+1) \\ \gamma_s(S) \end{bmatrix}. \quad (\text{A3})$$

On the other hand, in cases of $i = j = 1$ and $i = 1, j = S$, since $\gamma_S(1-S) = \gamma_1(S-1)$ by (44), we have

$$\begin{bmatrix} \gamma_1(S-1) & \gamma_1(0) \\ \gamma_S(0) & \gamma_1(S-1) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \gamma_1(S) \\ \gamma_S(1) \end{bmatrix}. \quad (\text{A4})$$

(A3) and (A4) are the YW equations in the general case. Since $\Sigma B^\top ((I - A)^{-1})^\top = \Gamma(1)^\top$, we have by (23) that

$$\Sigma = A\Sigma + B\Gamma(1)^\top + \Sigma_{\mathbf{M}} ((I - A)^{-1})^\top.$$

If Σ_{ε} is a diagonal matrix, then $\Sigma_{\mathbf{M}} ((I - A)^{-1})^\top$ is an upper triangular matrix. Thus, for each $s = 2, \dots, S$, we have the equation $\Sigma_{s,s-1} = \alpha_s \Sigma_{s-1,s-1} + \beta_s \Gamma(1)_{s-1,s}$, which is equivalent to $\gamma_{s-1}(1) = \alpha_s \gamma_{s-1}(0) + \beta_s \gamma_s(S-1)$. By replacing the first equation in (A3) by this equation, we see that $(\alpha_s, \beta_s)^\top$ is a solution of (46) for all $s = 2, \dots, S$. Now, using that $\gamma_{kS+s} = \gamma_s$ for all $k \in \mathbb{Z}$ and $s = 1, \dots, S$, we have $\gamma_0 = \gamma_S$. Then, by inverting the first and second rows, (A4) is equivalent to

$$\begin{bmatrix} \gamma_0(0) & \gamma_1(S-1) \\ \gamma_1(S-1) & \gamma_1(0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \gamma_0(1) \\ \gamma_1(S) \end{bmatrix}$$

which is (46) for $s = 1$. □

B Histograms of Yule-Walker estimates

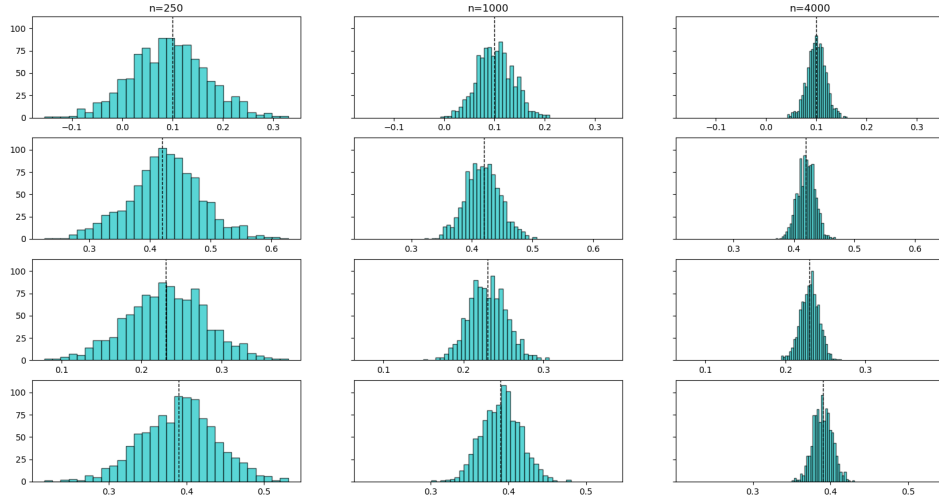


Figure 1: Histograms of YW estimates of parameters α_s 's for different sample size n generated with Sim1. (Dotted line denotes the true parameter value.)

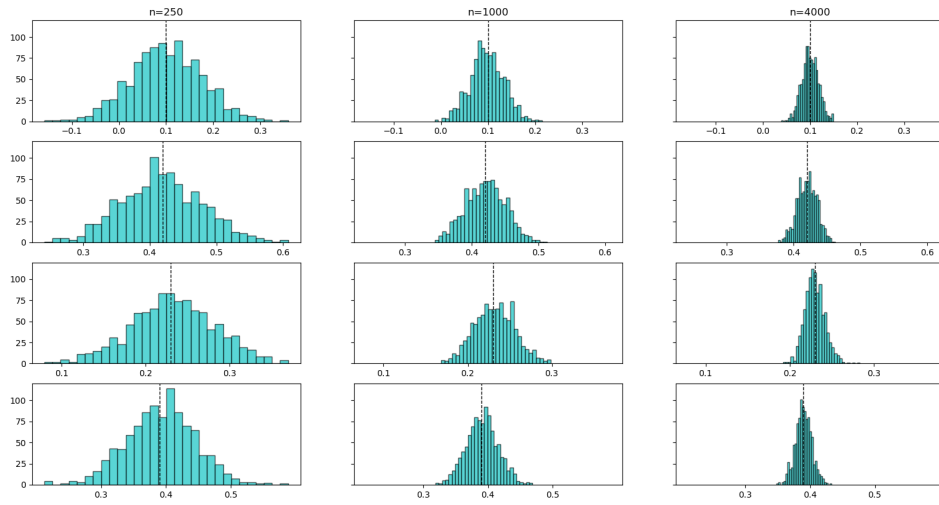


Figure 2: Histograms of YW estimates of parameters α_s 's for different sample size n generated with Sim2. (Dotted line denotes the true parameter value.)

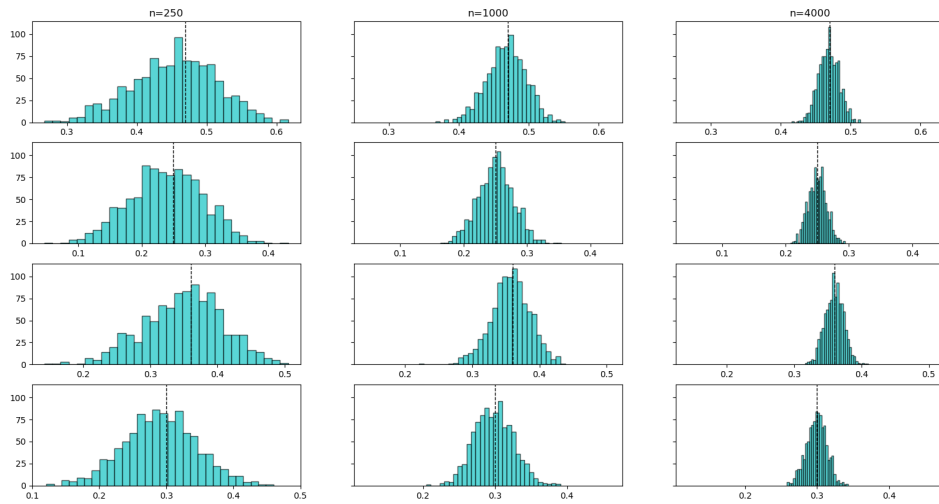


Figure 3: Histograms of YW estimates of parameters β_s 's for different sample size n generated with Sim1. (Dotted line denotes the true parameter value.)

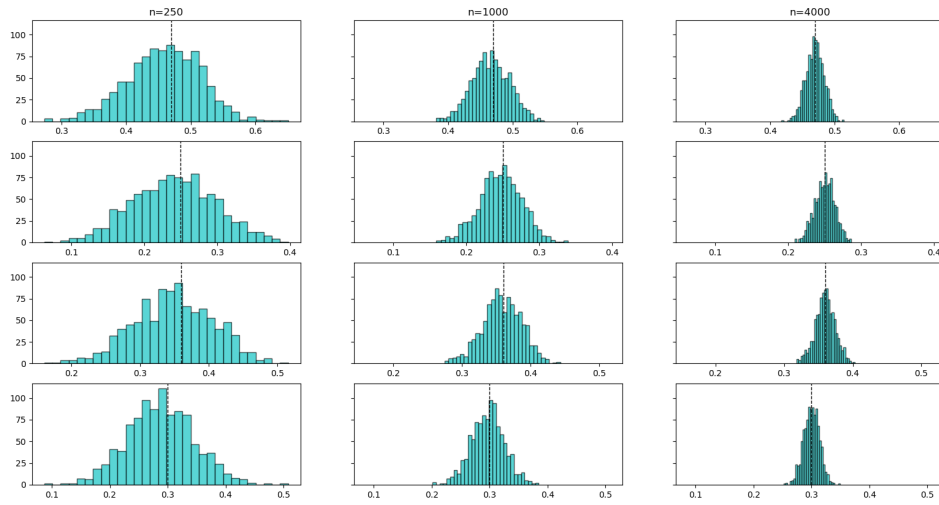


Figure 4: Histograms of YW estimates of parameters β_s 's for different sample size n generated with Sim2. (Dotted line denotes the true parameter value.)

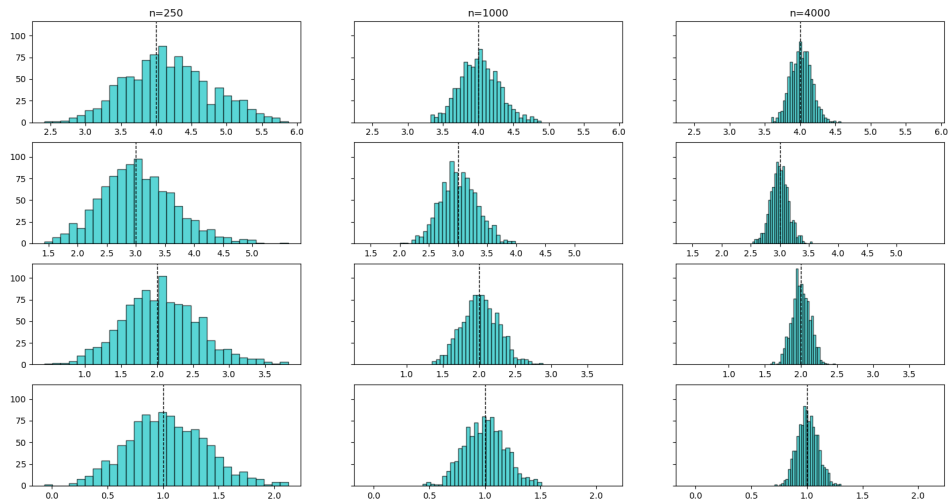


Figure 5: Histograms of YW estimates of parameters λ_s 's for different sample size n generated with Sim1. (Dotted line denotes the true parameter value.)

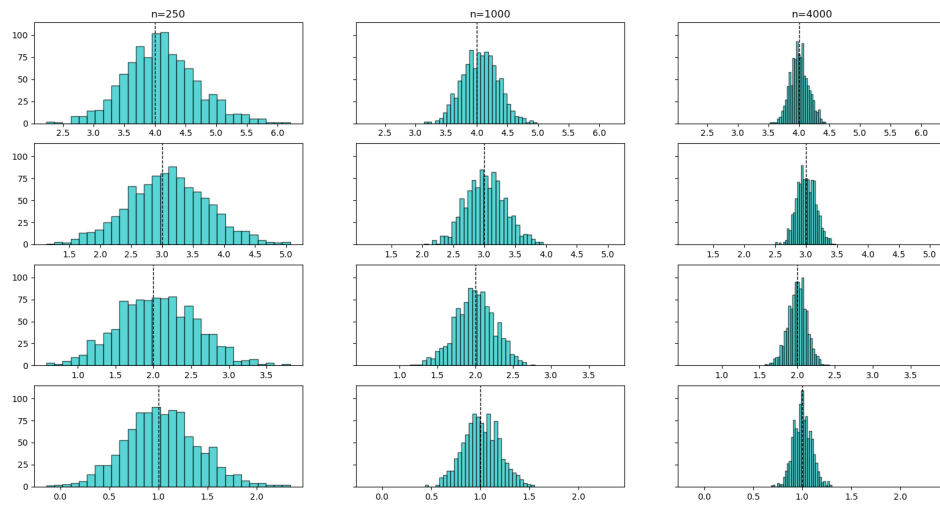


Figure 6: Histograms of YW estimates of parameters λ_s 's for different sample size n generated with Sim2. (Dotted line denotes the true parameter value.)

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