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Localized modeling of uncertainty in the Arlequin framework

R. Cottereau, D. Clouteau, and H. Ben Dhia

Abstract This paper discusses the coupling and interaction of a classical continuum model with another continuum model with random parameters. The former model, deterministic, aims at representing a part of the domain where the local fluctuations of the parameters, such as Young's modulus, do not influence the output of interest in a significant manner, and where a homogenized model is sufficient to predict this output. The latter model, stochastic, stands for the part of the domain where the local behavior is of interest and the fluctuations of the parameters cannot be considered only in a homogenized way. The coupling of these models is performed in the Arlequin framework. This paper focuses on the technical definitions of the spaces and operators introduced in the Arlequin framework for that particular problem, and on the definition of the corresponding discretized formulations. A simple example is shown, emphasizing the gain in computational power to compute the mean and confidence intervals in the region of interest.

1 Introduction

Classical deterministic models provide global predictions that are satisfactory for many industrial applications. However, when one is interested in a very localized behavior or quantity, or when multiscale phenomena come into play, these models

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may not be sufficient. For instance, the limited heterogeneity of a material modeled as a continuum might have no influence on its behavior on a large scale, while the study of a local stress intensity factor would strongly depend on the local heterogeneity of the mechanical parameters. Likewise, the prediction of the outbreak of a fracture in a structure might be performed with homogeneous models, while the incorporation of atomic modeling would be necessary to follow the exact path of that fracture. Unfortunately, for these problems, the information necessary to parameterize the relevant, very complex, models is usually not available. Stochastic methods have therefore been proposed and now appear unavoidable in multiscale modeling.

Although the use of stochastic models and methods has expanded rapidly in the last decades, the related numerical costs are still often prohibitive. Hence, the application of these methods in a complex or industrial context remains limited. An important field of research is therefore concerned with the reduction of the costs associated with the use of stochastic methods, for example by using iterative methods specially adapted to the structure of the matrices arising in the Stochastic Finite Element (FE) method [11, 14], using reduced bases for the representation of random fields [9], or using special domain decomposition techniques for parallel resolution on clusters of computers [16].

The present paper proposes an alternative to these purely mathematical/numerical approaches through the coupling of two models: one deterministic and one stochastic. The general goal is that of modeling a global problem in a mean or homogeneous way where it yields sufficient accuracy, while retaining a stochastic model where needed. Hence, additional complexity is added in the model only where required, and the general approach is both more elegant and numerically cheaper than a global all-over stochastic model would be. Further, the cuts on computational costs mean that industrial applications come within reach.

More specifically, we discuss here the interaction and coupling of a classical continuum model with another continuum model with random parameters. The former model, deterministic, aims at representing a part of the domain where the local fluctuations of the parameters, such as Young's modulus, do not influence the output of interest in a significant manner, and where a homogenized model is sufficient to predict this output. The latter model, stochastic, stands for the part of the domain where the local behavior is of interest and the fluctuations of the parameters cannot be considered only in a homogenized way. The coupling of these models is performed in the Arlequin framework [2, 4, 5, 6, 3]. Note that the choice of two continuous models is by no means a restriction of the contents of this paper, and that the Arlequin method can accommodate different models [8]. However, considering similar models allows us to concentrate more particularly on the specific aspects of the coupling of a deterministic model with a stochastic one.

The framework of this paper is very different from that of classical micromechanics [24, 7] and homogenization [13]. In these, the objective is to find a mean, or homogenized, behavior for a material, that will allow its study on a higher scale. In our case, we wish to study the local behavior of a small subdomain of that material, while the influence of the rest of the domain is taken into account in some homogenized way. Even when homogenization techniques are embedded within a stochastic

FE framework, with both scales actually represented, the coupling does not go both ways, and only the low scale influences the high one (see for instance [21]). This type of one-way coupling approach is also very classical in climate modeling [22], where the influence of the small, unresolved, details at the global scale are taken into account through stochastic models more or less tuned on a lower scale. Nevertheless, we will re-use some notions explored extensively in homogenization techniques. In particular the notion of size of a Representative Volume Element, with respect to the correlation length of the parameters of the medium and/or the number of realizations of that medium (see in particular [12, 13, 20, 23]) will be discussed in relation with the size of the coupling zone between our two models.

This work is closely related to previous works in the literature [18, 8]. However, in [18], the theoretical basis, different from the Arlequin formulation, is less general. In particular, it is only aimed at coupling a deterministic Boundary Element method with a stochastic FE method. In the recent work [8], the authors aim at coupling two stochastic models, one continuous, and one atomistic. However, many theoretical questions are left out. In particular, the coupling is performed between realizations of the stochastic operators, while we try to describe here the coupling at the level of the stochastic operators. The definition of the coupling operators is explicit, and the question of stability of the mixed problem can therefore be discussed (this will be done in a forthcoming paper). Also, it opens up the possibility of choosing among different numerical schemes, while the approach in [8] is limited to the Monte Carlo technique.

2 The classical Arlequin method

We first recall the Arlequin framework in the classical case of the coupling of two deterministic continuum models, with different meshes [2, 4, 5, 3]. This will allow to introduce more gently the relevant material for the coupling of a deterministic and a stochastic continuum models, which is of interest in this paper. Further, this will emphasize the novelty of the work presented here, and will make it easier for the reader already at ease with the Arlequin method.

Let us consider a domain Ω of \mathbb{R}^d , with smooth boundary $\partial\Omega$ separated into Dirichlet and Neumann boundaries Γ_D and Γ_N , such that $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $\Gamma_D \neq \emptyset$ (figure 1, left). We consider Poisson's equation, with a parameter K_0 , considered here constant, a bulk loading field $f(\mathbf{x})$, defined on Ω , and a surface loading field $\mathbf{g}(\mathbf{x})$, defined on Γ_N . Assuming for simplicity that the Dirichlet boundary condition is homogeneous, the weak formulation for this problem reads: find $\underline{u}_0 \in \mathcal{V}_0$ such that

$$\underline{a}_0(\underline{u}_0, v) = \ell_0(v), \forall v \in \mathcal{V}_0, \quad (1)$$

where $\underline{a}_0 : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}$ and $\ell_0 : \mathcal{V}_0 \rightarrow \mathbb{R}$ are defined, respectively, by $\underline{a}_0(u, v) = \int_{\Omega} K_0 \nabla u \cdot \nabla v \, d\Omega$, and $\ell_0(v) = \int_{\Omega} f v \, d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{n} v \, d\Gamma$, and

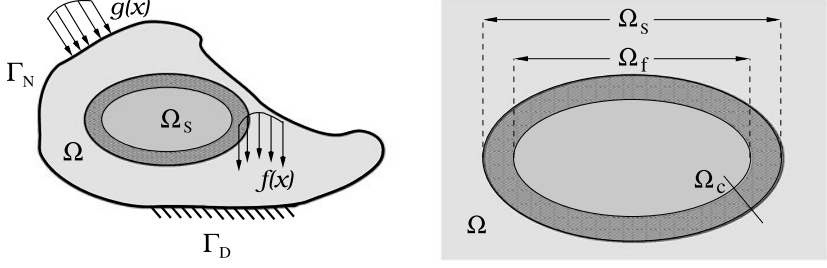


Fig. 1: Description of the model problem (left), and zoom on the zone where the two models are superposed (right), with the definition of the different domains: the homogenized domain Ω and the stochastic patch Ω_s , itself partitioned into a free zone Ω_f and a coupling zone $\Omega_c = \Omega_s \setminus \Omega_f$

$$\mathcal{V}_0 = \{v \in \mathcal{H}^1(\Omega), v|_{\Gamma_D} = 0\}.$$

This problem can be shown to have a unique solution \underline{u}_0 , which may, for example, be approximated by the Finite Element method.

Depending on the problem at hand, for example when there is a localized defect, it may be interesting to consider two very different meshes. A patch $\Omega_s \subset \Omega$ is therefore selected around the defect, and further partitioned into a free zone Ω_f and a coupled zone Ω_c , with $\Omega_f \cap \Omega_c = \emptyset$, and $\Omega_f \cup \Omega_c = \Omega_s$. A coarse finite element basis (supported by a coarse mesh) will be used on Ω to account for large scale deformations and stresses, and a refined one (supported on a fine mesh) will be used on Ω_s to reproduce more accurately the local effects around the defect. The Arlequin method allows to couple these two problems through the resolution of the following mixed problem: find $(u_0, u_s, \Phi) \in \mathcal{V}_0 \times \mathcal{V}_s \times \mathcal{V}_c$ such that

$$\begin{cases} a_0(u_0, v) + \underline{C}(\Phi, v) = \ell_0(v), & \forall v \in \mathcal{V}_0 \\ a_s(u_s, v) - \underline{C}(\Phi, v) = 0, & \forall v \in \mathcal{V}_s, \\ \underline{C}(\Psi, u_0 - u_s) = 0, & \forall \Psi \in \mathcal{V}_c \end{cases}, \quad (2)$$

where the bilinear functions $a_0 : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}$ and $a_s : \mathcal{V}_s \times \mathcal{V}_s \rightarrow \mathbb{R}$ are weighted restrictions of \underline{a}_0 , defined, respectively, by $a_0(u, v) = \int_{\Omega} \alpha_0 K_0 \nabla u \cdot \nabla v \, d\Omega$ and $a_s(u, v) = \int_{\Omega_s} \alpha_s K_0 \nabla u \cdot \nabla v \, d\Omega$, the coupling functional $\underline{C} : \mathcal{V}_c \times \mathcal{V}_c \rightarrow \mathbb{R}$ is defined by

$$\underline{C}(u, v) = \int_{\Omega_c} (\kappa_0 uv + \kappa_1 \nabla u \cdot \nabla v) \, d\Omega, \quad (3)$$

with κ_0 and κ_1 two constants (see for example [5] for details), the weights are chosen such that

$$\begin{cases} \alpha_0, \alpha_s \geq 0 & \text{in } \Omega \\ \alpha_0 + \alpha_s = 1 & \text{in } \Omega \\ \alpha_0 = 1 & \text{in } \Omega/\Omega_s \\ \alpha_0, \alpha_s \text{ constant} & \text{in } \Omega_f \end{cases}, \quad (4)$$

and the functional spaces are $\mathcal{V}_s = \{v \in \mathcal{H}^1(\Omega_s)\}$ and $\mathcal{V}_c = \{v \in \mathcal{H}^1(\Omega_c)\}$.

Note that, for simplicity, this has been derived in the case when the patch Ω_s is totally included inside the domain Ω . In particular, the patch does not intersect the Dirichlet boundary condition, *i.e.* $\partial\Omega_s \cap \Gamma_D = \emptyset$, and the loads are outside the patch, *i.e.* that is $f(\mathbf{x} \in \Omega_s) = 0$. However, more general results can be obtained without any further theoretical difficulty [3]. Several important propositions have been derived in different papers and summed up and completed in [3]. We recall two such propositions below, in the simplified case considered, and without the corresponding proofs.

Proposition 1 (Stability). *Under classical regularity hypotheses on the domains $\Omega_0, \Omega_s, \Omega_c$ and Ω_f , with $\Omega_c \cup \Omega_f = \Omega_s$, $\Omega_c \cap \Omega_f = \emptyset$, and $\text{meas}(\Omega_s) \neq 0$, assuming the hypotheses (4) on the weights, and supposing $K_0 > 0$, the Arlequin problem (2) admits a unique solution (u_0, u_s, Φ) in $\mathcal{V}_0 \times \mathcal{V}_s \times \mathcal{V}_c$.*

Proposition 2 (Consistency). *Under the hypotheses (4) on the weights, the solution u_{arl} of the Arlequin problem (2), defined by*

$$u_{arl} = \begin{cases} u_0 & \text{in } \Omega_0 \setminus \Omega_s \\ \alpha_0 u_0 + \alpha_s u_s & \text{in } \Omega_s \end{cases},$$

verifies

$$u_{arl} = \underline{u}_0,$$

where \underline{u}_0 is the unique solution of the monomodel reference problem (1). Further, if the restriction to Ω_f of the displacement field solution is regular, we also have

$$u_0 = u_s = \underline{u}_0 \text{ in } \Omega_f.$$

3 The continuous stochastic-deterministic Arlequin formulation

In this section, we introduce the main result of this paper, *i.e.* the mixed formulation for the coupling of a stochastic continuous model with a deterministic continuous model. This is an extension of the results described in the previous section, and the main novelties are in the choice of the coupling and lift operators. We start by introducing the stochastic monomodel, where the physical parameter is modeled as stochastic on the entire domain Ω , and then move on to the coupled formulation.

3.1 The stochastic monomodel

Let us therefore consider that the physical parameter is modeled by a random field $\mathbf{K}(\mathbf{x}) \in \mathcal{L}^2(\Theta, \mathbb{R})$, where (Θ, \mathcal{F}, P) is a complete probability space, with Θ a set of outcomes, \mathcal{F} a σ -algebra of events, and $P: \mathcal{F} \rightarrow [0, 1]$ a probability measure. We suppose (as in [1] for example) that this field is bounded and uniformly coercive, that is to say $\exists K_{\min}, K_{\max} \in (0, +\infty)$, such that $P(\mathbf{K}(\mathbf{x}) \in [K_{\min}, K_{\max}], \forall \mathbf{x} \in \Omega) = 1$. The weak formulation of the corresponding stochastic boundary value problem now reads: find $\underline{\mathbf{u}}_0 \in \mathcal{W}_0$ such that

$$\underline{\mathcal{A}}_0(\underline{\mathbf{u}}_0, \mathbf{v}) = \mathcal{L}_0(\mathbf{v}), \forall \mathbf{v} \in \mathcal{W}_0 \quad (5)$$

where $\underline{\mathcal{A}}_0: \mathcal{W}_0 \times \mathcal{W}_0 \rightarrow \mathbb{R}$ and $\mathcal{L}_0: \mathcal{W}_0 \rightarrow \mathbb{R}$ are defined, respectively, by $\underline{\mathcal{A}}_0(\mathbf{u}, \mathbf{v}) = \mathbb{E}[\int_{\Omega} \mathbf{K} \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\Omega]$, and $\mathcal{L}_0(\mathbf{v}) = \int_{\Omega} f \mathbb{E}[\mathbf{v}] d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{n} \mathbb{E}[\mathbf{v}] d\Gamma$, $\mathbb{E}[\cdot]$ denotes the mathematical expectation, and

$$\mathcal{W}_0 = \mathcal{L}^2(\Theta, \mathcal{V}_0).$$

The above hypothesis on the parameter field $\mathbf{K}(\mathbf{x})$ ensures that the stochastic bilinear form $\underline{\mathcal{A}}_0$ is continuous and coercive on $\mathcal{W}_0 \times \mathcal{W}_0$. The Lax-Milgram theorem can therefore be used [1] to prove the existence and uniqueness of the solution $\underline{\mathbf{u}}_0$. An approximation of this solution can be obtained, for example, by using a Stochastic FE method [10, 17] or a Monte Carlo method [15].

Note that the requirement of boundedness of the mechanical parameter field is a quite stringent one. Another set of conditions has been derived in [19] that relaxes the upper bound property, while still preserving the solvability of the associated stochastic boundary value problem.

3.2 The Arlequin formulation

We now wish to superpose, in the Arlequin framework, two models: one deterministic, in Ω ; and one stochastic, in Ω_s . We will therefore consider two models of the parameter field: a deterministic one, K_0 , supposed constant on the domain Ω ; and a stochastic one, $\mathbf{K}(\mathbf{x})$, modeled as a random field on Ω_s . We further suppose that $\mathbf{K}(\mathbf{x})$ verifies the conditions described above, and that $\mathbb{E}[\mathbf{K}(\mathbf{x})] = K_0$. The stochastic-deterministic Arlequin problem reads: find $(u_0, \mathbf{u}_s, \Phi) \in \mathcal{V}_0 \times \mathcal{W}_s \times \mathcal{W}_c$ such that

$$\begin{cases} a_0(u_0, v) + C(\Phi, v) = \ell_0(v), & \forall v \in \mathcal{V}_0 \\ \mathcal{A}_s(\mathbf{u}_s, \mathbf{v}) - C(\Phi, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathcal{W}_s, \\ C(\Psi, u_0 - \mathbf{u}_s) = 0, & \forall \Psi \in \mathcal{W}_c \end{cases} \quad (6)$$

where the bilinear forms $a_0: \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{R}$, $\mathcal{A}_s: \mathcal{W}_s \times \mathcal{W}_s \rightarrow \mathbb{R}$, and $C: \mathcal{W}_c \times \mathcal{W}_c \rightarrow \mathbb{R}$ are defined by

$$a_0(u, v) = \int_{\Omega} \alpha_0 K_0 \nabla u \cdot \nabla v \, d\Omega,$$

$$\mathcal{A}_s(\mathbf{u}, \mathbf{v}) = \mathbb{E} \left[\int_{\Omega_s} \alpha_s \mathbf{K} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega \right],$$

and

$$C(\mathbf{u}, \mathbf{v}) = \mathbb{E} \left[\int_{\Omega_c} (\kappa_0 \mathbf{u} \mathbf{v} + \kappa_1 \nabla \mathbf{u} \cdot \nabla \mathbf{v}) \, d\Omega \right],$$

the fields $\alpha_0(\mathbf{x})$ and $\alpha_s(\mathbf{x})$ verify the conditions (4), and the functional spaces \mathcal{W}_s and \mathcal{W}_c are given by

$$\mathcal{W}_c = \{ \underline{\mathbf{v}}(\mathbf{x}) + \theta \mathbb{I}_c(\mathbf{x}) | \underline{\mathbf{v}} \in \mathcal{H}^1(\Omega_c), \theta \in \mathcal{L}^2(\Theta, \mathbb{R}), \mathbb{E}[\theta] = 0 \},$$

and

$$\mathcal{W}_s = \mathcal{L}^2(\Theta, \mathcal{H}^1(\Omega_s)),$$

and where the indicator function $\mathbb{I}(\mathbf{x})$ is such that $\mathbb{I}_c(\mathbf{x} \in \Omega_c) = 1$ and $\mathbb{I}_c(\mathbf{x} \notin \Omega_c) = 0$. Note that the space \mathcal{W}_c can be seen as composed of functions with a spatially varying mean and perfectly spatially correlated randomness.

Note that, thanks to the specific structure of the space \mathcal{W}_c , the last line of the system (6) can be written equivalently,

$$\begin{aligned} C(\Psi, u_0 - \mathbf{u}_s) &= \mathbb{E} \left[\int_{\Omega_c} (\kappa_0 (\underline{\Psi} + \theta \mathbb{I}_c)(u_0 - \mathbf{u}_s) + \kappa_1 \nabla \underline{\Psi} \cdot \nabla (u_0 - \mathbf{u}_s)) \, d\Omega \right] \\ &= \underline{C}(\mathbb{E}[\Psi], u_0 - \mathbb{E}[\mathbf{u}_s]) + \mathbb{E} \left[\theta \int_{\Omega_c} (u_0 - \mathbf{u}_s) \, d\Omega \right] \\ &= 0, \quad \forall \Psi \in \mathcal{W}_c. \end{aligned}$$

Therefore, this condition imposes that the mean of the field \mathbf{u}_s should be equal to the field u_0 , in all points of Ω_c , and that the variability of $\int_{\Omega_c} (\mathbb{E}[\mathbf{u}_s] - \mathbf{u}_s) \, d\Omega$ should cancel. In other words, this means that some degree of homogenization takes place within the coupling zone. In particular, if that zone is not big enough with respect to the correlation lengths of the fields $\mathbf{K}(\mathbf{x})$ or $\mathbf{u}_s(\mathbf{x})$, the Arlequin scheme is expected to yield results that would be different from those obtained with the stochastic monomodel. It would mean that there is not enough localization of the variability and stochasticity for the Arlequin scheme to make sense.

The next section addresses the issue of the discretization of the mixed formulation to obtain computable estimates of the different solution fields.

4 The discretized stochastic-deterministic Arlequin formulation

The general idea here is to use a classical FE approach to discretize the first equation of the Arlequin system (6) and a stochastic FE method for the second and third ones. It should be noted that a Monte Carlo resolution of this formulation is not

straightforward because the coupling equation (the third one) works on the mean of \mathbf{u}_s , which is not available when one considers only one realization of that random field. A modified scheme to solve this problem using of Monte Carlo simulations will be described in a forthcoming paper.

We therefore associate to the domain Ω a mesh \mathcal{T} , composed of elements E , to the domain Ω_s a mesh \mathcal{T}_s , composed of elements E_s , and to the domain Ω_c a mesh \mathcal{T}_c , composed of elements E_c . We look for approximate functions of the elements of \mathcal{V}_0 , \mathcal{V}_s and \mathcal{V}_c in the functional spaces

$$\mathcal{V}_0^H = \{v \in P_1(E), v|_{\Gamma_D} = 0\},$$

$$\mathcal{V}_s^H = \{v \in P_1(E_s)\},$$

and

$$\mathcal{V}_c^H = \{v \in P_1(E_c)\},$$

composed of linear functions on each of the elements of the meshes. We then choose the bases $\{v_\ell^0(\mathbf{x})\}_{1 \leq \ell \leq m_0}$, $\{v_\ell^s(\mathbf{x})\}_{1 \leq \ell \leq m_s}$, and $\{v_\ell^c(\mathbf{x})\}_{1 \leq \ell \leq m_c}$ for the functions in \mathcal{V}_0^H , \mathcal{V}_s^H , and \mathcal{V}_c^H , respectively. We introduce the matrices A_0 , C_0 , and C_s , with elements

$$A_{0,ij} = \int_{\Omega_0} \alpha_0 K_0 \nabla v_i^0 \cdot \nabla v_j^0 d\Omega,$$

$$C_{0,ij} = \int_{\Omega_c} (\kappa_0 v_i^0 v_j^c + \kappa_1 \nabla v_i^0 \cdot \nabla v_j^c) d\Omega,$$

and

$$C_{s,ij} = \int_{\Omega_c} (\kappa_0 v_i^s v_j^c + \kappa_1 \nabla v_i^s \cdot \nabla v_j^c) d\Omega.$$

For the space \mathcal{W}_s , we choose an approximating space as the span of the polynomial chaos basis [10], of order n and degree p , in conjunction with the previous basis for the spatial dimension. We denote this space $\mathcal{W}_s^{H,n,p}$ and the elements of its basis $\{w_{k\ell}^s(\mathbf{x}) = v_k^s(\mathbf{x}) \hat{\Gamma}_\ell\}_{1 \leq k \leq m_s, 0 \leq \ell \leq N-1}$, where N is the number of elements in the polynomial chaos basis, which depends both on n and p . We approximate both the parameter field $\mathbf{K}(\mathbf{x})$, and the solution $\mathbf{u}_s(\mathbf{x})$ in that basis,

$$\mathbf{u}_s(\mathbf{x}) \approx \sum_{k=1}^{m_s} \sum_{\ell=0}^{N-1} u_s^{k\ell} v_k^s(\mathbf{x}) \hat{\Gamma}_\ell,$$

and finally obtain the matrix \mathbb{A} for the stochastic part of the Arlequin system as

$$\mathbb{A}_{j\ell, JL} = \sum_{i=1}^N c_{ijJ} \int_{\Omega_s} \alpha_s(\mathbf{x}) k_i(\mathbf{x}) \nabla v_\ell^s(\mathbf{x}) \cdot \nabla v_L^s(\mathbf{x}) d\Omega,$$

where $c_{ijJ} = \mathbb{E}[\hat{\Gamma}_i[\xi] \hat{\Gamma}_j[\xi] \hat{\Gamma}_J[\xi]]$ and $k_i(\mathbf{x}) = \mathbb{E}[\mathbf{K}(\mathbf{x}) \hat{\Gamma}_i(\xi)]$. Note that the double indices (j, ℓ) and (J, L) each correspond to only one index in the matrix form of the system. We denote by \mathbf{M} the $m_s \times (N-1)$ matrix with general term

$$M_{k\ell} = \delta_\ell \int_{\Omega_c} v_k^s(\mathbf{x}) d\Omega,$$

where the δ_ℓ is related to the vector ordering of the coefficients $\mathbf{u}_{k\ell}^s$.

We further denote

$$\underline{\mathbb{A}} = \mathbb{A}_{0\ell,0L} = \int_{\Omega_0} \alpha_s K_0 \nabla v_i^s \cdot \nabla v_j^s d\Omega,$$

and $\mathbb{A}_c = \mathbb{A}_{0\ell,JL}$ the sub-matrix that corresponds to the coupling of the mean part of the unknown field with the fluctuating part, and $\mathbb{A}_s = \mathbb{A}_{j\neq 0 \ell, J\neq 0L}$ that corresponding to the fluctuating part. We finally get the form of the matrix system for the Arlequin problem (6):

$$\begin{bmatrix} A_0 & 0 & 0 & C_0 & 0 \\ 0 & \underline{\mathbb{A}} & \mathbb{A}_c & -C_s & 0 \\ 0 & \underline{\mathbb{A}}^T & \mathbb{A}_s & 0 & M \\ C_0^T & -C_s^T & 0 & 0 & 0 \\ 0 & 0 & M^T & 0 & 0 \end{bmatrix} \begin{bmatrix} U_0 \\ \underline{U} \\ U_s \\ \Phi \\ \Lambda \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (7)$$

where the coordinates of the vector F are defined by $F_i = \ell_0(v_i)$, U_0 , \underline{U} , U_s , and Φ are the vectors of coordinates of u_0 , $E[\mathbf{u}_s]$, $\mathbf{u}_s - E[\mathbf{u}_s]$, and Φ , in the bases of \mathcal{V}_0^H , \mathcal{V}_s^H , $\mathcal{W}_s^{H,n,p} \setminus \mathcal{V}_s^H$, and \mathcal{V}_c^H , respectively, and Λ is the vector of the Lagrange multipliers enforcing the homogenization in the coupling zone. Note that the controlling parameters for the size of that matrix are n and p , and that in most cases, \mathbb{A}_s will be a very large matrix, much larger than the other ones appearing in the system (7). However, it is much smaller than the matrix that would be obtained by applying directly a stochastic FE approach to the entire model.

5 Example of application

For illustrative purposes, we consider the indented domain of Fig. 2, with $-3 < x < 3$ and $-1 < y < 1$, with a neck in the zone around $x = 0$, loading $f(x, y) = 1$ in the zone $2.5 < x < 3$ (right side of the plate), and homogeneous Dirichlet boundary conditions on the left side of the plate. The boundary conditions read:

$$\begin{cases} u = 0, & x = -3 \\ \nabla u = \mathbf{0}, & \text{on } \partial\Omega \setminus \{x = -3\} \end{cases}$$

The resolution of this problem with a homogeneous parameter $K_0 = 1$, using a FE scheme based on Eq. (1) leads to the intensity field $|K_0 \nabla u|$ represented in Fig. 3. In this figure, we also plot the point-wise mean and the point-wise 90%-confidence interval of the intensity field $|\mathbf{K} \nabla \mathbf{u}|$ of the corresponding stochastic problem given in Eq. (5). For that problem, the parameter field is modeled as a random field $\mathbf{K}(\mathbf{x})$

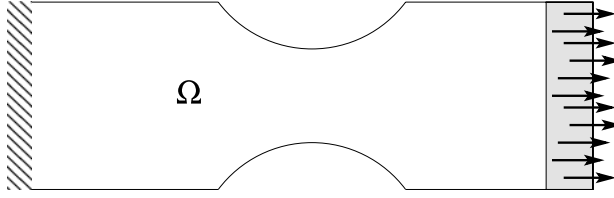


Fig. 2: Scheme of the model problem: indented domain Ω , homogeneous Dirichlet boundary conditions on the left side, and bulk forces in an area on the right side

as in [19], with a mean $E[\mathbf{K}(\mathbf{x})] = K_0 = 1$, a dispersion parameter $\delta = 0.2$ and a correlation length $\ell_c = 0.15$ m.

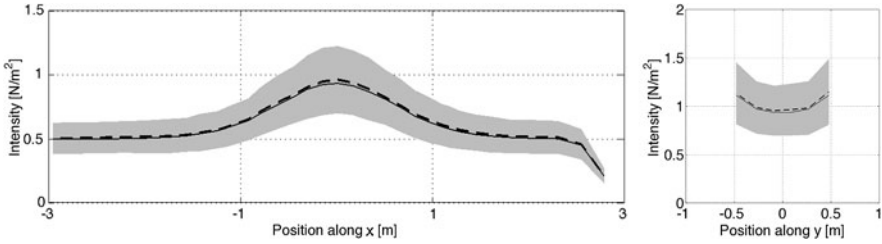


Fig. 3: Intensity field $|\mathbf{K}\nabla\mathbf{u}|$ corresponding to the solution of the deterministic monomodel (Eq. (1)) (solid line), and of the stochastic monomodel, Eq. (5) (the dashed line indicates the point-wise mean and the grey shade indicates the point-wise 90%-confidence interval)

Finally, we solve the corresponding coupled Arlequin problem with a region of interest Ω_s contained in the zone $-1.7 < x < 1.7$, and with a coupling region Ω_c of length 0.6 on each side (*i.e.* contained within $-1.7 < x < -1.1$ and $1.1 < x < 1.7$). We obtain the results plotted in Fig. 4, where it can be seen that the coupled Arlequin problem gives the same solution as the full heterogeneous problem in the region of interest. Note also that, at the limit of the coupling region, the approximation obtained from the Arlequin method seems to deteriorate compared to the full scale simulation. As the ensemble mean and the space mean are intricately woven in our approach, this was only to be expected. However, if the coupling zone is large enough, this effect does not contaminate the solution in the free zone Ω_f . Further numerical studies on the convergence of this coupled approach, depending on the size of the coupling zone and the correlation length, will be performed in future work.

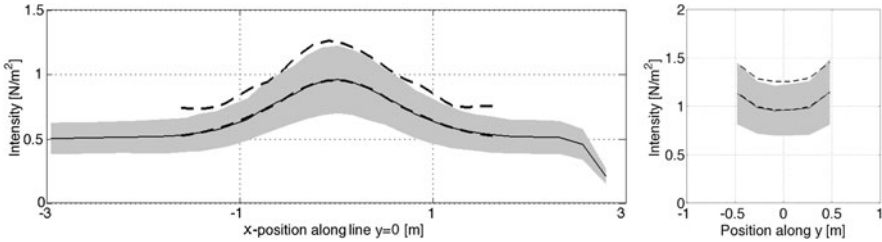


Fig. 4: Intensity field $|\mathbf{KV}\mathbf{u}|$ corresponding to the solution of the stochastic monomodel, Eq. (5) (the solid line indicates the point-wise mean and the grey shade indicates the point-wise 90%-confidence interval), and corresponding values obtained through the resolution of the coupled Arlequin problem of Eq. (6) (dashed lines for the mean and upper bound of the 90%-confidence interval)

6 Conclusion

We have shown here a method for coupling a probabilistic model of continuum mechanics with a deterministic one. The numerical costs associated with the resolution of a probabilistic model are heavily lowered, which renders its use in an industrial setting reasonable. The framework that was described here can very easily be extended to other problems, be it with different physics (continuum mechanics, molecular dynamics, nonlinear constitutive relation), using the available literature on the Arlequin method, or involving two probabilistic models. For the latter, a discussion will be necessary to choose the adequate coupling operator. In an upcoming paper, we will discuss further the particularities of the method, and in particular: the stability of the mixed formulation, the size of the coupling zone with respect to the definition of representative volume elements in homogenization techniques [12, 13, 20, 23], and the possibility to solve the coupled system using the Monte Carlo simulation technique.

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