



HAL
open science

Further insights by theoretical investigations of the multiscale Arlequin method

Hachmi Ben Dhia

► **To cite this version:**

Hachmi Ben Dhia. Further insights by theoretical investigations of the multiscale Arlequin method. International Journal for Multiscale Computational Engineering, 2008, 6 (3), pp.215-232. 10.1615/IntJMultCompEng.v6.i3.30 . hal-04704826

HAL Id: hal-04704826

<https://centralesupelec.hal.science/hal-04704826v1>

Submitted on 21 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Further insights by theoretical investigations of the multiscale Arlequin method

Hachmi Ben Dhia

Ecole Centrale Paris, MSS-MAT, UMR 8579 CNRS, 92295 Chatenay Malabry CEDEX, France

ABSTRACT

The Arlequin method allows for concurrent multimodel and multiscale analyses of mechanical problems. This method relies on a number of parameters, the rigorous choice of which being an essential issue. Some analyses are carried out in this article and are shown to be helpful for the design of reliable choices. These theoretical investigations are enlightened by simple but relevant numerical tests.

KEYWORDS

multimodel, multiscale, partition of models, Arlequin method

*Address all correspondence to hachmi.ben-dhia@ecp.fr

1. INTRODUCTION

To adapt locally in a given zone S of a given global mechanical model that is inappropriate in S , the Arlequin methodology [1–4] proceeds as follows:

- superpose in S a local appropriate model to the global locally inappropriate one
- define a partition of energies of the superposed models by using weight parameter functions
- couple the models in a part of S , named the “gluing” or the “coupling” zone.

The resulting Arlequin model is then obtained as a partition of the superimposed models. A prototype is shown in Fig. 1, where a fine 2-D cracked finite element model is locally superimposed and glued in the yellow zone to an uncracked global coarse 2-D finite element model to simulate (hopefully with enhanced modeling flexibility) a global cracked plate. The Arlequin method has been assessed numerically for the superposition and coupling of continuum-continuum models [1–10]. We also refer to [11–17] for the coupling of atomistic and continuum models by using approaches taking a cue from or having similarities with the Arlequin method. However, this method involves many parameters: the superposition zone and its partition into a gluing and a free zone, the coupling operator, the Lagrange multiplier space (called the *mediator* space), and the weight parameter functions. The choice of a reliable set of these parameters is an essential issue. Some choices have already been designed formally [1,2]

or through first mathematical stability results [6,18]. The latter were established under relatively stringent conditions on the weight parameter functions that do not allow, for instance, for the consideration of regular weight functions such as the ramp functions used in [11] and [12]. Recently, by revisiting and extending numerical works that can be found in [4], a discussion concerning the L^2 and H^1 coupling operators and the weight parameter functions was reported in [10]. However the L^2 coupling is misinterpreted in [4] and [10], leading to a misinterpretation of the numerical results obtained with the finite element L^2 coupling and consequently to coarse comparisons between the L^2 and H^1 coupling operators. Moreover, in all the works reported on the Arlequin method, no precise definition is given of the superposition zone, the coupling zone, and the free zone. Finally, though it is intuitively expected that by stressing the local altered model, the Arlequin solution would be near the solution of the monomodel altered mechanical problem, and though this intuition seems to be numerically well founded, there is still a lack of precise and optimal hypotheses on the parameters under which a rigorous proof can be established for what can be called the relevance of the Arlequin framework.

The main topic of the article is to carry out some analyses of continuous and discrete finite element Arlequin problems that provide further insights on choices of reliable Arlequin parameters. For this, the linear elasticity problem is formulated in the Arlequin framework by considering different Lagrange multiplier-based coupling operators, and the obtained mixed Arlequin problems are first

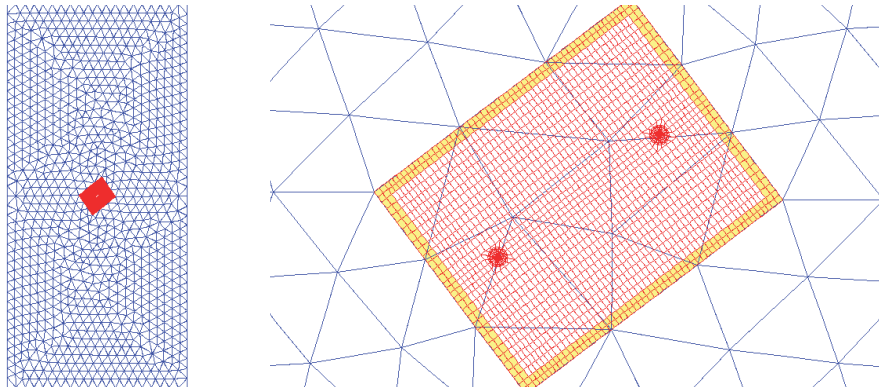


FIGURE 1. A typical Arlequin model: a cracked plate is superimposed locally to a sound global plate. The right part is a zoom around the local cracked model, showing the mesh refinement in the neighborhood of the crack tips

discussed in Section 2. Of particular interest is the interpretation of the discrete L^2 coupling as an approximation of the continuous duality coupling (and not the continuous L^2 coupling). This interpretation enlightens the finite element Lagrange multiplier irregularity, noticed first in [4] (with discontinuous weight functions) and in [10] (Fig. 10) for continuous weight functions, when an L^2 coupling is used. Section 2 ends with a consistency result. In Section 3, we derive continuous and discrete stability results for the mixed Arlequin problems under suitable hypotheses on the parameters of the approach. Concerning these results, let us underline two salient points. First, we extend to the case of regular partitions of the internal energy the proofs of the stability of the mixed continuous and discrete finite element Arlequin problems, established for discontinuous partitions in [18], giving, in this way, a rigorous assessment of the use of regular weight functions such as ramp functions. Second, we clarify the very local loss of stability that may occur when coarse and fine finite element models are superposed and when the space of Lagrange multipliers is taken to be equal to the restriction of the gluing zone of the coarse finite element space. In Section 4, we prove theretically the relevance of the Arlequin framework.

Numerical results are given in Section 5 to assess our theoretical results, and the article closes with a summary of our major conclusions concerning practical choices of the Arlequin method parameters. Some perspectives and ongoing works are also given.

2. SOME ARLEQUIN FORMULATIONS OF THE ELASTICITY PROBLEM

To give simple and comprehensive derivations and analyses of Arlequin problems, we consider the following representative model.

2.1 The Classical Linearized Elasticity Problem

Let us consider an elastic body occupying the closure of a bounded regular domain Ω_1 included in \mathfrak{R}^d , with $d = 1, 2$, or 3 in practice (observe that here and in the sequel, the regularity of a domain will refer to a Lipschitz continuity of its boundary). It is submitted to a field of volume density of forces

$\mathbf{f} \in \mathbf{L}^2(\Omega_1)$ and clamped on Γ_u , a nonzero measured part of its boundary $\partial\Omega_1$ when $d = 2$ or $d = 3$. The remaining Γ_g of the boundary is assumed to be free.

The weak primal (*monomodel*) formulation of this problem reads as follows (e.g., [7]):

$$\begin{aligned} & \text{find } \mathbf{u}_0 \in \mathbf{W}_0 \\ & \forall \mathbf{v}_0 \in \mathbf{W}_0, \quad a_0(\mathbf{u}_0, \mathbf{v}_0) = l_0(\mathbf{v}_0) \end{aligned} \quad (1)$$

where the admissible displacement field space \mathbf{W}_0 and the virtual works of the inner and outer forces, respectively, read

$$\mathbf{W}_0 = \{\mathbf{v}_0 \in \mathbf{H}^1(\Omega_1) \mid \mathbf{v}_0 = \mathbf{0} \text{ on } \Gamma_u\} \quad (2)$$

$$\begin{aligned} & \forall (\mathbf{u}_0, \mathbf{v}_0) \in \mathbf{W}_0 \times \mathbf{W}_0 \\ & a_0(\mathbf{u}_0, \mathbf{v}_0) = \int_{\Omega_1} \boldsymbol{\sigma}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{v}_0) \end{aligned} \quad (3)$$

$$\forall \mathbf{v}_0 \in \mathbf{W}_0, \quad l_0(\mathbf{v}_0) = \int_{\Omega_1} \mathbf{f} \cdot \mathbf{v}_0 \quad (4)$$

In these equations, $\boldsymbol{\varepsilon}(\mathbf{v}_0)$ and $\boldsymbol{\sigma}(\mathbf{v}_0)$ denote the linearized strain and stress tensors, associated with field \mathbf{v}_0 , which are assumed to be connected through Hooke's law:

$$\sigma_{ij} = R_{ijkl} \varepsilon_{kl} \quad (5)$$

where i, j, k , and l range from 1 to d and the convention of summation over repeated indices is used. The elasticity moduli R_{ijkl} are assumed to satisfy classical regularity, symmetry, and *coercivity* hypotheses:

$$R_{ijkl} \in L^\infty(\Omega_1) \quad (6)$$

$$R_{ijkl} = R_{jikl} = R_{klij} \quad (7)$$

$$\exists c > 0; \forall \tau_{ij} = \tau_{ji}, \quad R_{ijkl} \tau_{ij} \tau_{kl} \geq c \tau_{ij} \tau_{kl} \text{ in } \Omega_1 \quad (8)$$

It is well known that, under the hypotheses listed previously, the problem defined by (1)–(5) is well posed.

This model problem will now be formulated in the Arlequin framework without, in a first step, introducing any alteration.

2.2 Lagrange Multiplier-Based Arlequin Formulations of the Elasticity Problem

Let Ω_2 be a nonzero measured given bounded regular domain overlapping Ω_1 . Let Ω_{12} be the overlap

(denoted by S in Section 1). For clarity and with no major restrictions, it will be assumed that Ω_2 is strictly embedded in Ω_1 , leading to $\Omega_{12} = \Omega_2$ (situation depicted by Fig. 1). The overlap is partitioned into two regular nonoverlapping domains, i.e.,

$$\Omega_{12} = \Omega_{12}^c \cup \Omega_{12}^f \quad (9)$$

where Ω_{12}^c is the models gluing zone and Ω_{12}^f is the free zone. The former is assumed to satisfy the following hypotheses (see the yellow zone of the zoom in Fig. 1):

$$\partial\Omega_{12} \subset \partial\Omega_{12}^c \quad (10)$$

$$\text{meas}(\Omega_{12}^c) > 0 \quad (11)$$

In the latter, the models are superimposed but not coupled to each other. They are *free* from each other (see the pink zone of the zoom in Fig. 1). It will also be assumed in the sequel that Ω_{12} , Ω_{12}^c , and, subsequently, Ω_{12}^f are given (adaptivity is needed otherwise).

Now, to define Arlequin formulations of the elasticity problem, we denote by \mathbf{W}_1 and \mathbf{W}_2 the following spaces of kinematical admissible fields:

$$\mathbf{W}_1 = \{\mathbf{v}_1 \in \mathbf{H}^1(\Omega_1) ; \mathbf{v}_1 = \mathbf{0} \text{ on } \Gamma_u\} \quad (12)$$

$$\mathbf{W}_2 = \mathbf{H}^1(\Omega_2) \quad (13)$$

Weighted internal and external virtual works are defined by

$$\begin{aligned} \mathbf{u}_1, \mathbf{v}_1 &\in \mathbf{W}_1 \\ a_1(\mathbf{u}_1, \mathbf{v}_1) &= \int_{\Omega_1} \alpha_1 \boldsymbol{\sigma}(\mathbf{u}_1) : \boldsymbol{\varepsilon}(\mathbf{v}_1) \end{aligned} \quad (14)$$

$$\begin{aligned} \mathbf{u}_2, \mathbf{v}_2 &\in \mathbf{W}_2 \\ a_2(\mathbf{u}_2, \mathbf{v}_2) &= \int_{\Omega_2} \alpha_2 \boldsymbol{\sigma}(\mathbf{u}_2) : \boldsymbol{\varepsilon}(\mathbf{v}_2) \end{aligned} \quad (15)$$

$$\mathbf{v}_1 \in \mathbf{W}_1, \quad l_1(\mathbf{v}_1) = \int_{\Omega_1} \beta_1 \mathbf{f} \cdot \mathbf{v}_1 \quad (16)$$

$$\mathbf{v}_2 \in \mathbf{W}_2, \quad l_2(\mathbf{v}_2) = \int_{\Omega_2} \beta_2 \mathbf{f} \cdot \mathbf{v}_2 \quad (17)$$

where weight parameter functions (α_1, β_1) and (α_2, β_2) are defined, respectively, in Ω_1 and Ω_2 and satisfy (see Fig. 2)

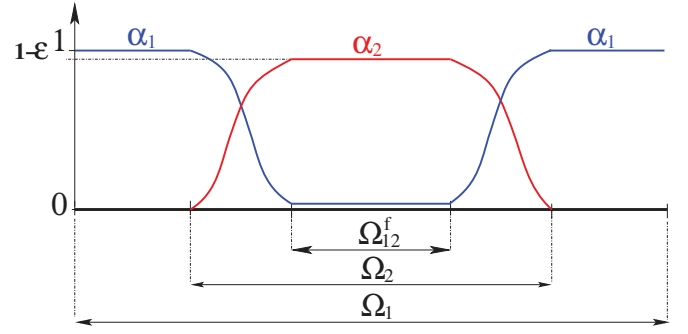


FIGURE 2. Weight parameter functions $\alpha_i, i = 1, 2$

$$\begin{aligned} \alpha_i &\geq 0 \quad \text{in } \Omega_i, \quad \alpha_i = 1 \quad \text{in } \Omega_i/\Omega_{12} \\ \alpha_1 + \alpha_2 &= 1 \quad \text{in } \Omega_{12} \end{aligned} \quad (18)$$

$$\begin{aligned} \beta_i &\geq 0 \quad \text{in } \Omega_i, \quad \beta_i = 1 \quad \text{in } \Omega_i/\Omega_{12} \\ \beta_1 + \beta_2 &= 1 \quad \text{in } \Omega_{12} \end{aligned} \quad (19)$$

Observe that with the hypotheses (18) and (19), these functions are bounded.¹

2.2.1 Dual Volume Coupling

The dual volume coupling–based continuous Arlequin formulation of the model elasticity problem reads [2]

$$\begin{aligned} \text{find } (\mathbf{u}_1, \mathbf{u}_2, \boldsymbol{\Phi}_d) &\in \mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{M}_d \\ \forall (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\Psi}) &\in \mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{M}_d \\ a_1(\mathbf{u}_1, \mathbf{v}_1) + C_d(\boldsymbol{\Phi}_d, \mathbf{v}_1) &= l_1(\mathbf{v}_1) \end{aligned} \quad (20)$$

$$a_2(\mathbf{u}_2, \mathbf{v}_2) - C_d(\boldsymbol{\Phi}_d, \mathbf{v}_2) = l_2(\mathbf{v}_2) \quad (21)$$

$$C_d(\boldsymbol{\Psi}, \mathbf{u}_1 - \mathbf{u}_2) = 0 \quad (22)$$

where \mathbf{W}_1 , \mathbf{W}_2 , a_1 , a_2 , l_1 , and l_2 are defined by (12) to (17), respectively, with the weight parameters verifying (18) and (19). The (dual) mediator space is denoted by \mathbf{M}_d . It is the dual space of $\mathbf{M} = \mathbf{W}_1/\Omega_{12}^c = \mathbf{W}_2/\Omega_{12}^c = \mathbf{H}^1(\Omega_{12}^c)$. The volume coupling operator, denoted by C_d , reads

$$\forall (\boldsymbol{\Psi}, \mathbf{v}) \in \mathbf{M}_d \times \mathbf{M}, \quad C_d(\boldsymbol{\Psi}, \mathbf{v}) = \langle \boldsymbol{\Psi}, \mathbf{v} \rangle_{\mathbf{M}_d, \mathbf{M}} \quad (23)$$

where $\langle \cdot, \cdot \rangle_{\mathbf{M}_d, \mathbf{M}}$ stands for the duality bracket.

¹ Other couples of weight parameter functions have to be introduced if one takes into account other types of energy or virtual mechanical works such as virtual works of inertial forces in dynamic regimes, and so on.

The dual volume coupling is a natural mechanical coupling operator in the sense that, interpreting the Lagrange multiplier field Φ_d as a density of forces, it has to be in the dual space M_d of the space of the displacements in Ω_{12}^c . However, with this operator, the finite element approximation of Eqs. (20)–(23) is an intricate issue. This is due to the subtlety of elements of M_d . Actually, for the discrete finite element approximation of the Eqs. (20)–(23), the operator C_d can be approximated by an $L^2(\Omega_{12}^c)$ scalar product. But, unless appropriately scaled [9], this L^2 -coupling operator would lead to an ill conditioning of the associated discrete Arlequin problems (as shown in the numerical section).

A suitable coupling alternative is defined in the next subsection.

2.2.2 Energy Coupling

By using the classical Riesz-Fréchet representation theorem (e.g., [20]), the natural H^1 -scalar product of the space $M = \mathbf{H}^1(\Omega_{12}^c)$ can be substituted into the duality bracket. For the model elasticity problem, a second mixed Arlequin problem using an equivalent energy scalar product can then be derived. It reads as follows:

$$\text{find } (\mathbf{u}_1, \mathbf{u}_2, \Phi) \in \mathbf{W}_1 \times \mathbf{W}_2 \times M$$

$$\forall \mathbf{v}_1 \in \mathbf{W}_1, \quad a_1(\mathbf{u}_1, \mathbf{v}_1) + C(\Phi, \mathbf{v}_1) = l_1(\mathbf{v}_1) \quad (24)$$

$$\forall \mathbf{v}_2 \in \mathbf{W}_2, \quad a_2(\mathbf{u}_2, \mathbf{v}_2) - C(\Phi, \mathbf{v}_2) = l_2(\mathbf{v}_2) \quad (25)$$

$$\forall \Psi \in M, \quad C(\Psi, \mathbf{u}_1 - \mathbf{u}_2) = 0 \quad (26)$$

where the coupling operator, denoted by $C(\cdot, \cdot)$, is now defined by

$$\forall (\Psi, \mathbf{v}) \in M \times M$$

$$C(\Psi, \mathbf{v}) = \int_{\Omega_{12}^c} \kappa_0 \{ \Psi \cdot \mathbf{v} \} + \kappa_1 \{ \boldsymbol{\epsilon}(\Psi) : \boldsymbol{\epsilon}(\mathbf{v}) \} \quad (27)$$

The positive parameters κ_0 and κ_1 are homogeneous to a string rigidity divided by a length and to a material rigidity, respectively.

Observe here that while the (dual) Lagrange multiplier Φ_d appearing in (20)–(23) is homogeneous to a volume density of forces that controls the deviation between \mathbf{u}_1 and \mathbf{u}_2 in Ω_{12}^c , the Lagrange multiplier Φ appearing in (24)–(27) is homogeneous to

a displacement field. This field can be understood as the (primal) Lagrange multiplier that controls the deviation between forces in Ω_{12}^c . Actually, this remark gives precisely a mechanical interpretation of the Riesz representation theorem.

2.3 Consistency of the Arlequin Framework

In the Arlequin problems, defined previously, the superposed models use the same linearized elasticity model. It is then natural to require that the solution \mathbf{u}^0 of the monomodel elasticity problem, defined by (1)–(5), be recovered through the Arlequin solutions. This consistency result is proved here. For this, we anticipate the stability results of the next section and assume that the Arlequin problems are well posed.

Proposition 2.1 *Under the hypotheses (18) and (19) and by assuming that the weight parameters $\alpha_i, i = 1, 2$, are constant in Ω_{12}^f , the solutions of the Arlequin problems, defined by (20)–(23) and (24)–(27), verify*

$$\mathbf{u}^{arl} = \mathbf{u}^0 \quad \text{in } \Omega_1 \quad (28)$$

where \mathbf{u}^0 is the unique solution of the monomodel reference elasticity problem, defined by (1)–(5) and where \mathbf{u}^{arl} is defined by

$$\mathbf{u}^{arl} = \begin{cases} \mathbf{u}_1 & \text{in } \Omega_1/\Omega_{12} \\ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 & \text{in } \Omega_{12} \end{cases} \quad (29)$$

Moreover, if one assumes that $\alpha_i = \beta_i, i = 1, 2$, and that the restriction to the free zone Ω_{12}^f of the displacement field solutions are regular, then we also have

$$\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}^0 \quad \text{in } \Omega_{12}^f \quad (30)$$

Proof. Let us give the proof for the solution of the Arlequin problem (24)–(27) since exactly the same proof can be carried out for the solution of the other problem.

Observe that if $(\mathbf{u}_1, \mathbf{u}_2)$ is the displacement part of the solution of problems (24)–(27), then, thanks to (26), (27), and (11), the Arlequin displacement, defined by (29), is an element of \mathbf{W}_0 .

Let now \mathbf{v}_0 be any admissible field in \mathbf{W}_0 . Take \mathbf{v}_1 equal to the restriction of \mathbf{v}_0 to Ω_1 in (24). Take \mathbf{v}_2 equal to the restriction of \mathbf{v}_0 to Ω_2 in (25). This gives two equations. The result given by (28) is then obtained by adding these two equations, using

(26), (27), the fact that the α_i are constant in Ω_{12}^f (as shown by Fig. 2), and the uniqueness of the solution of the linearized elasticity problem.

The second part of the proposition is a consequence of the fact that, under a sufficient regularity of \mathbf{u}^0 , \mathbf{u}_1 and \mathbf{u}_2 , in Ω_{12}^f , one can check easily (by using once more the fact that the α_i are constant in Ω_{12}^f) that, in Ω_{12}^f , the three fields are governed by the same local linearized elasticity problem, with the same prescribed (\mathbf{u}^0) Dirichlet boundary condition on $\partial\Omega_{12}^f$. A uniqueness of solution argument for the latter problem ends, then, the proof of the proposition.

Before analyzing the stability of the Arlequin problems defined previously, let us make here some remarks:

1. The coupling operator $C(.,.)$, defined by (27), can be replaced by any other scalar product that is equivalent to the natural energy scalar product. One choice consists of using the following expression:

$$\forall(\Psi, \mathbf{v}) \in M \times M \quad (31)$$

$$C(\Psi, \mathbf{v}) = \int_{\partial S_u} \kappa_2 \{ \Psi \cdot \mathbf{v} \} + \int_{\Omega_{12}^c} \kappa_1 \{ \boldsymbol{\varepsilon}(\Psi) : \boldsymbol{\varepsilon}(\mathbf{v}) \}$$

where κ_2 and κ_1 are positive parameters homogeneous to a string and a material rigidity, respectively, and where ∂S_u is a nonzero measured part of $\partial\Omega_{12}$.

2. The practical importance of the Arlequin framework is actually recovered in a second step: Since the Arlequin formulations, defined previously, involve two superposed mechanical models, different approximations and/or alterations of these models are potentially possible. For instance, different finite element approximations, behavior laws, structure modeling, material cohesion, and so on can be used. This is precisely the prime goal of the Arlequin framework.
3. For the static Arlequin problems defined previously, the models are coupled basically by enforcing weakly the equality between the displacement fields in Ω_{12}^c . For more complex superposed models, such as atomistic and continuum models, it has been suggested to equate

weakly in Ω_{12}^c a transformation of these displacement fields by means of interpolation, averaging, or mollifying operators (which can be called the *accommodation* operators). We refer to [1,2] for first considerations of this aspect and to [13,14,17] for examples in which a discrete 1-D atomistic displacement field is transformed to a continuum one by means of an interpolation operator (using a linear finite element basis and the atoms as nodes), before being coupled to the continuum displacement. Moreover, an approximation of the atomistic energy by a sum of (1) the continuum energy evaluated at the accommodated atomistic field in an appropriate part of the gluing zone and (2) the atomistic energy for the atoms located in the remainder of the superposition zone (containing the free zone) seems helpful for the achievement of an effective atomistic-continuum coupling (this is an ongoing work).

3. STABILITY OF THE ARLEQUIN PROBLEMS

In this section, we analyze the stability of the Arlequin problems, defined in the previous section, and give through our investigations indications on the way one can choose the parameters of the approach. Of particular practical interest are the proofs of stability of the continuous and discrete finite element mixed Arlequin problems, under less stringent conditions on the internal weight functions than in [18].

3.1 Some Hypotheses and Notations

In addition to (18), the couple of internal weight parameter functions is assumed to satisfy

$$\forall i \in \{1, 2\}, \exists \alpha_0 > 0; \alpha_i \geq \alpha_0, \text{ in } \Omega_{12}^f \quad (32)$$

Observe that the condition (32) is less stringent than the one considered in [18], namely,

$$\forall i \in \{1, 2\}, \exists \alpha_0 > 0; \alpha_i \geq \alpha_0, \text{ in } \Omega_{12} \quad (33)$$

The difference between (32) and (33) is important from a practical point of view since (32) allows for the use of regular couples of weight functions α_i , $i = 1, 2$, such as the ramp function plotted in Fig. 2, while (33) does not. Indeed, with (33), α_1 has to drop from 1 in the outside of Ω_{12} to at most $1 - \alpha_0$

when crossing $\partial\Omega_{12}$. We will refer to (32) and (33) as the regular and irregular α -weight conditions, respectively.

The stability results given in the sequel rely on the Babuška and Brezzi theories [21,22] for mixed problems. To cope with these theories, the following notations and definitions are helpful. For a given regular and bounded domain ω included in \mathfrak{R}^d and for an integer $m \geq 0$, classical Sobolev spaces (e.g., [23]) are denoted by $\mathbf{H}^k(\omega)$. In these Hilbert spaces, classical norms are denoted by $\|\cdot\|_{m,\omega}$, for m ranging from 0 to m . We define now the following product space of kinematically admissible fields:

$$\mathbf{W} = \mathbf{W}_1 \times \mathbf{W}_2 \quad (34)$$

equipped with the norm $\|\cdot\|_{\mathbf{W}}$:

$$\forall \mathbf{v} \in \mathbf{W}, \quad \|\mathbf{v}\|_{\mathbf{W}}^2 = \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\mathbf{v}_2\|_{1,\Omega_2}^2 \quad (35)$$

We also define the following global bilinear and linear (obviously) continuous forms:

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{W}, \quad a(\mathbf{u}, \mathbf{v}) = a_1(\mathbf{u}_1, \mathbf{v}_1) + a_2(\mathbf{u}_2, \mathbf{v}_2) \quad (36)$$

$$\forall \mathbf{v} \in \mathbf{W}, \quad l(\mathbf{v}) = l_1(\mathbf{v}_1) + l_2(\mathbf{v}_2) \quad (37)$$

with a_1, a_2, l_1 , and l_2 defined by (14) to (17), respectively.

Finally, for each coupling operator C and for each subspace \mathbf{m} of the (primal) mediator space \mathbf{M} , we define the kernel, with respect to \mathbf{m} , of C denoted by $\mathbf{N}_m(C)$:

$$\mathbf{N}_m(C) = \{ \mathbf{v} \in \mathbf{W}; \forall \Psi \in \mathbf{m}, C(\Psi, \mathbf{v}_1 - \mathbf{v}_2) = 0 \} \quad (38)$$

3.2 Stability Results for the Arlequin Problems

The well-posedness of the previously defined Arlequin problems is shown in this subsection.

3.2.1 Analysis of the Duality Coupling-Based Arlequin Problem

The main result follows.

Proposition 3.1 *Under classical regularity hypotheses on the domains $\Omega_1, \Omega_{12}, \Omega_{12}^c$, and Ω_{12}^f , the hypotheses (9)–(11) on the overlapping and coupling domains, the hypotheses (6)–(8) on the elasticity moduli, and the hypotheses (18),(19), and (32) on the weight parameter*

functions, the problem defined by (20)–(23), with (5) and (12)–(17), admits a unique solution (\mathbf{u}, Φ_d) in $\mathbf{W} \times \mathbf{M}_d$.

Proof. The considered problem is a linear mixed problem for which mathematical theories are available. We use the theory of Brezzi [22] (quite linked to the one of Babuška [21]). To cope with this theory, one has to check some properties for the bilinear and linear forms defining problems (20)–(23). The two main properties (the others being obvious) are the *coercivity* of the bilinear form a on $\mathbf{N}_{M_d}(C_d)$ and the *Inf-Sup* (or *LBB*) condition.

- *Coercivity:* Though it is possible to derive the same result in more general situations, for the sake of clarity, it is assumed here that α_2 is equal to 0 in Ω_{12}^c . Moreover, here and in the sequel, and unless mentioned explicitly, c denotes a strictly positive constant, independent of the field $\mathbf{v} \in \mathbf{W}$.

Now, for all \mathbf{v} in $\mathbf{N}_{M_d}(C_d)$, using the properties (6)–(8) of the elasticity moduli, a Korn's inequality, and the definition of $\mathbf{N}_{M_d}(C_d)$ implying the equality of the fields \mathbf{v}_1 and \mathbf{v}_2 in Ω_{12}^c , one can easily deduce that

$$a(\mathbf{v}, \mathbf{v}) \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_2)\|_{0,\Omega_2}^2 \} \quad (39)$$

But since \mathbf{v}_1 and \mathbf{v}_2 are equal in Ω_{12}^c , one can deduce from (39) (with a different positive constant c , of course) that

$$a(\mathbf{v}, \mathbf{v}) \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_2)\|_{0,\Omega_2}^2 + \|\mathbf{v}_2\|_{1,\Omega_{12}^c}^2 \} \quad (40)$$

Thanks to the hypothesis $\text{meas}(\Omega_{12}^c) > 0$, a second use of Korn's inequality allows us to conclude from (40), *coercivity* of the bilinear form a .

- *Inf Sup condition:* By definition, the norm of a field Ψ in \mathbf{M}_d is given by

$$\|\Psi\|_{\mathbf{M}_d} = \sup_{\substack{\mathbf{v} \in \mathbf{M} \\ \|\mathbf{v}\|_{\mathbf{M}} \neq 0}} \frac{\langle \Psi, \mathbf{v} \rangle_{\mathbf{M}_d, \mathbf{M}}}{\|\mathbf{v}\|_{1,\Omega_{12}^c}} \quad (41)$$

For the bilinear form C_d , we have

$$\begin{aligned} \forall \Psi \in M_d, \quad & \sup_{\substack{\mathbf{v} \in \mathbf{W} \\ \|\mathbf{v}\|_{\mathbf{W}} \neq \mathbf{0}}} \frac{C_d(\Psi, \mathbf{v}_1 - \mathbf{v}_2)}{\|\mathbf{v}\|_{\mathbf{W}}} \\ & \geq \sup_{\substack{\mathbf{v}_2 \in \mathbf{W}_2 \\ \|\mathbf{v}_2\|_{\mathbf{W}_2} \neq \mathbf{0}}} \frac{C_d(\Psi, \mathbf{v}_2)}{\|\mathbf{v}_2\|_{1, \Omega_2}} \end{aligned} \quad (42)$$

Referring to (41) and (42), the existence of a continuous extension operator from $H^1(\Omega_{12}^c)$ into $H^1(\Omega_2)$, relying on the assumed regularity of Ω_{12}^c (see, e.g., [24]), allows us to end the proof of the *Inf-Sup* condition and thus the proof of the proposition.

3.2.2 Analysis of the Energy Scalar Product-Based Arlequin Problems

By using exactly the same arguments as those developed in the poof of proposition 3.1, we can establish the following existence and uniqueness result.

Proposition 3.2: Stability of the H^1 -coupling Arlequin problem. *Under the hypotheses of proposition 3.1, the problem defined by (24)–(27), with (5) and (12)–(17), admits a unique solution (\mathbf{u}, Φ) in $\mathbf{W} \times M$.*²

3.2.3 Stability of Discrete Arlequin Problems

Let us here recall a discrete stability result given in [18]. Assume that the domains Ω_1 , Ω_2 and Ω_{12}^c are polygonal in 2-D and polyheral in 3-D. Consider regular triangulations of Ω_1 and Ω_2 and define conformal linear finite element spaces \mathbf{W}_{h_1} , \mathbf{W}_{h_2} , and $\mathbf{W}_h = \mathbf{W}_{h_1} \times \mathbf{W}_{h_2}$ (see, e.g., [25,26]). For the construction of the finite element (primal) mediator space M_{h_t} , we assume for simplicity that Ω_{12}^c is composed of elements of the mesh of Ω_2 (yellow zone of Fig. 1),

$$M_{h_t} = \mathbf{W}_{h_2} | \Omega_{12}^c \quad (43)$$

or Ω_{12}^c is composed of elements of the mesh of Ω_1

² If, in problems (20)–(23), one replaces the duality coupling by an L^2 coupling, the obtained problem could be meaningless. The numerical example given in the left part of Fig. 5 shows clearly that when an $L^2(\Omega_{12}^c)$ coupling is used, with constant weight functions α_i , the Lagrange multiplier is not an L^2 field.

$$M_{h_t} = \mathbf{W}_{h_1} | \Omega_{12}^c \quad (44)$$

where $\mathbf{W}_{h_i} | \Omega_{12}^c$ denotes the restriction of the space \mathbf{W}_{h_i} , $i = 1, 2$, to the coupling zone Ω_{12}^c . Moreover, we define the following discrete kernels of the coupling operator C :

$$N_m^h(C) = \{\mathbf{v} \in \mathbf{W}_h; \forall \Psi \in m, C(\Psi, \mathbf{v}_1 - \mathbf{v}_2) = 0\} \quad (45)$$

The family of discrete Arlequin problems, derived from the continuous H^1 scalar product-based Arlequin problem, reads

$$\text{find } (\mathbf{u}_{h_1}, \mathbf{u}_{h_2}, \Phi_{h_t}) \in \mathbf{W}_{h_1} \times \mathbf{W}_{h_2} \times M_{h_t}$$

$$\forall \mathbf{v}_{h_1} \in \mathbf{W}_{h_1}, a_1(\mathbf{u}_{h_1}, \mathbf{v}_{h_1}) + C(\Phi_{h_t}, \mathbf{v}_{h_1}) = l_1(\mathbf{v}_{h_1}) \quad (46)$$

$$\forall \mathbf{v}_{h_2} \in \mathbf{W}_{h_2}, a_2(\mathbf{u}_{h_2}, \mathbf{v}_{h_2}) - C(\Phi_{h_t}, \mathbf{v}_{h_2}) = l_2(\mathbf{v}_{h_2}) \quad (47)$$

$$\forall \Psi_{h_t} \in M_{h_t}, C(\Psi_{h_t}, \mathbf{u}_{h_1} - \mathbf{u}_{h_2}) = 0 \quad (48)$$

For each of these discrete Arlequin problems, we have the following:

Proposition 3.3 *Under the hypotheses of proposition 3.2, with (33) substituted into (32), and under the hypotheses listed previously on the domains Ω_1 , Ω_2 and Ω_{12}^c , the hypotheses on the meshes and on the discrete finite element spaces \mathbf{W}_{h_1} , \mathbf{W}_{h_2} , and M_{h_t} , each of the discrete problems, defined by (46)–(48), admits a unique solution $(\mathbf{u}_h, \Phi_{h_t}) \in \mathbf{W}_h \times M_{h_t}$. Moreover, we have an optimal convergence of the finite element solutions to the continuous Arlequin solution, i.e., $\exists c > 0$ (independent of h_1 , h_2 , and h_t):*

$$\begin{aligned} & \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{W}} + \|\Phi - \Phi_{h_t}\|_{1, \Omega_{12}^c} \\ & \leq c \left\{ \inf_{\mathbf{v}_h \in \mathbf{W}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{W}} + \inf_{\Psi_{h_t} \in M_{h_t}} \|\Phi - \Psi_{h_t}\|_{1, \Omega_{12}^c} \right\} \end{aligned} \quad (49)$$

where (\mathbf{u}, Φ) is the solution of the problem defined by (24)–(27).

The proof of this proposition, mainly the proofs of uniform discrete *coercivity* and *Inf-Sup* conditions, leading (see, e.g., [22]) to an optimal convergence of the discrete finite element solutions to the solution of the continuous Arlequin problem, are established in [18]. But let us give here some indications: under the irregular α -weight condition (33), one can establish easily (by following a very classical procedure) a uniform *coercivity* of a in the space $N_{\mathfrak{R}_{\Omega_{12}^c}}(C)$, where $\mathfrak{R}_{\Omega_{12}^c}$ denotes the space of rigid body modes of Ω_{12}^c . The uniform discrete *coercivity* condition of the bilinear form a in $N_{\mathfrak{R}_{\Omega_{12}^c}}^h(C)$ is then

an obvious consequence of the continuous *coercivity* and the assumed embedding of the discrete finite element space \mathbf{W}_h in \mathbf{W} . The proof of a uniform discrete *Inf-Sup* condition is less obvious. By following the proof of the *Inf-Sup* condition for proposition 3.1, it is achieved by constructing, under, say, the hypothesis (43), a discrete continuous extension operator from M_{ht} into \mathbf{W}_{h_2} . This most delicate part of the proof is carried out in several steps in [18]. These steps are briefly described in the following:

- M_{ht} and \mathbf{W}_{h_2} are vector-valued finite element spaces, but we observe that it is sufficient to construct a continuous extension operator by considering scalar finite element spaces we denote by M_{ht} and W_{h_2}
- For any given Ψ_h in M_{ht} , we associate the local Poisson problem $\Delta v = 0$, in Ω_{12}^f , with $v = \Psi_h$, on $\partial\Omega_{12}^f$
- We define then an operator from M_{ht} into W_{h_2} , by associating to Ψ_h in M_{ht} the element $v_h(\Psi_h)$ of W_{h_2} , whose restriction to Ω_{12}^c is equal to Ψ_h and whose restriction to Ω_{12}^f is equal to the finite element approximation of the solution of the variational formulation of the Poisson problem (defined in the previous step), by using the space of restriction of W_{h_2} to Ω_{12}^f
- By (1) using a regularity result for the solution of the Poisson problem with prescribed regular Dirichlet boundary conditions (see, e.g., [24,27]), (2) following by essence the technical developments of Widlund [28,29], and (3) using the continuity of the trace operator, we can check that the operator constructed in the previous step is a continuous extension operator from M_{ht} into W_{h_2} .

This ends the proof of the proposition.

3.2.4 Analysis of Discrete Arlequin Problems under Less Stringent Conditions

Internal weight parameter functions like ramp functions do not satisfy the condition (33). Hence, for a reliable use of such functions (which may be important in practice), we have to prove the stability of the discrete mixed Arlequin problems, under

the less stringent condition (32). Actually, referring to the proof of proposition 3.3, the main point that has to be rechecked is the discrete uniform *coercivity*. Let us address partially this issue, referring to Section 5.2 for a complementary enlightening numerical result (and to a forthcoming work for an extended analysis).

If the condition (33) is weakened to the condition (32), while keeping all the other hypotheses of proposition 3.3, then, by following exactly the lines as for the proof of the *coercivity* for proposition 3.1, a discrete uniform *coercivity* of a in $\mathbf{N}_{M_{ht}}^h(C)$, defined by (45), can be established easily if one assumes that the following *H-hypothesis* is satisfied: the finite element spaces \mathbf{W}_{h_1} , \mathbf{W}_{h_2} , and M_{ht} are such that the elements of the kernel space $\mathbf{N}_{M_{ht}}^h(C)$ verify $\mathbf{v}_{h_1} = \mathbf{v}_{h_2}$, in Ω_{12}^c .

One can easily check that the *H-hypothesis* is satisfied, for example, when the spaces $\mathbf{W}_{h_1} | \Omega_{12}^c$ and $\mathbf{W}_{h_2} | \Omega_{12}^c$ are compatible with each other, with $\mathbf{W}_{h_2} | \Omega_{12}^c$ finer than $\mathbf{W}_{h_1} | \Omega_{12}^c$ and $M_{ht} = \mathbf{W}_{h_2} | \Omega_{12}^c$ (the [primal] *mediator* space is equal to the finer displacement space). On the contrary, if one takes $M_{ht} = \mathbf{W}_{h_1} | \Omega_{12}^c$ (the [primal] *mediator* space is equal to the coarser space), then, depending on the choice of the α_i , the *coercivity* condition may become mesh-dependent or even be lost, leading to instabilities. The second example given in the numerical Section 5 enlightens this aspect (see also [10], Fig. 12, for the case of a H^1 coupling with a ramp weight function and a coarse multiplier space).

4. RELEVANCE OF THE ARLEQUIN FRAMEWORK

Until now, we have considered two similar superposed models to address the stability and consistency of the Arlequin problems. But as already mentioned at the end of Section 2, the fundamental interest of the Arlequin framework relies on the fact that the Arlequin formulations involve concurrently blended mechanical models. Different approximations and/or alterations of these blended models are thus potentially possible, leading to real multimodel (or multiscale) problems. The main theoretical issue is the analysis of the link that may exist between the multimodel Arlequin solution and the related *monomodel* mechanical solution. Fundamentally, the question is, What is the link between the classical (monomodel) solution of a linear elas-

tic problem modeling a cracked elastic solid and the solution of the problem modeling the same but uncracked (or sound) elastic solid, to which a local cracked solid is superimposed in the Arlequin framework (see Fig. 1), in a way that the cracks are located exactly in the same place?

As a final result of our article, we establish a limit behavior result for the family of the Arlequin solutions, indexed by the weight parameter functions. To achieve this task, we keep on considering the energy scalar product-based continuous Arlequin problem.

When two really different models are superimposed, the consistency issue addressed in proposition 2.1 is irrelevant. Indeed, in these situations, the Arlequin solutions depend strongly on the choice of the weight parameter functions. We refer to the results shown by Fig. 3, on which we calculate a 2-D elastic model, defined in a sound domain Ω_1 , on which we superimpose a local elastic model, defined in a cracked domain Ω_2 . A coarse bilinear finite element space is used to approximate the global solution, and a fine bilinear finite element space is used to approximate the local solution. The left edge of the structure is clamped, the right is free, and the two other edges are submitted to a traction load. We use the energy-based Arlequin formulation and we glue the two models away from the crack, in a zone whose boundary contains the boundary of the superposition zone. In Fig. 3(left), the local model

is stressed (α_2 , denoted by α_{fissure} in the figure, is taken to be constant and equal to 0.999 in the whole domain Ω_2), while in Fig. 3(left), the global sound model is stressed ($\alpha_{\text{fissure}} = 0.001$). One can notice the dramatic sensibility of the Arlequin solutions to the internal weight parameters.

Thus, when using the Arlequin framework to mix different models, an essential practical issue is the appropriate choice of these weight parameter functions. To our best understanding, and for some particular partitions of different models (deformable and rigid models, for instance), the appropriate choice of the α_i is trivial (see [3,9]), the limit behavior of the Arlequin solutions, whenever either the pair (α_1, β_1) or the pair (α_2, β_2) tends to $(0, 0)$ in Ω_{12}^f , is the most relevant practical issue. This is precisely the point of focus of this section.

Let us consider two superposed linear elastic models in the Arlequin framework. But let us now assume that in Ω_{12}^f the model defined in the domain Ω_2 , is geometrically altered, say, by a hole to fix the ideas, so that Ω_2 is transformed to $\Omega_{2a} = (\Omega_2/\Omega_{12}^f) \cup \Omega_{12}^{fa}$, where Ω_{12}^{fa} denotes the altered free zone. The model, defined in the domain Ω_1 , is not changed (see Fig. 8(left)). Moreover, we assume that the alteration is strictly embedded in the interior of Ω_{12}^f .

The new continuous altered H^1 -scalar product-based Arlequin problem reads

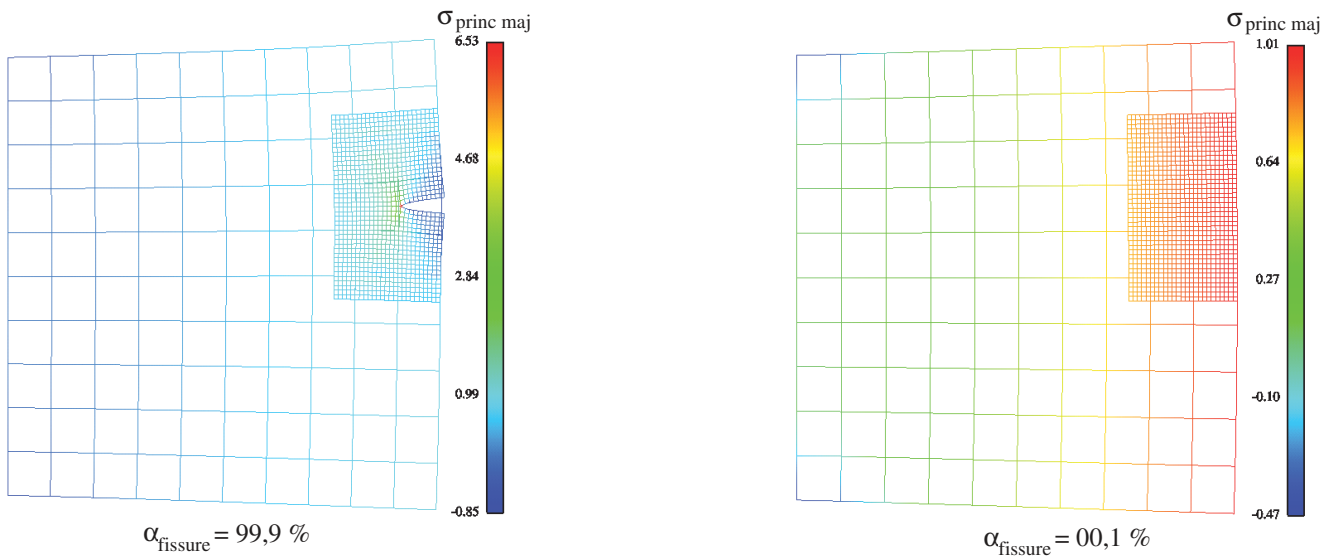


FIGURE 3. Sensibility of the Arlequin solution to the weight internal energy parameters

$$\text{find } (\mathbf{u}_1, \mathbf{u}_{2a}, \Phi) \in \mathbf{W}_1 \times \mathbf{W}_{2a} \times M$$

$$\forall \mathbf{v}_1 \in \mathbf{W}_1, \quad a_1(\mathbf{u}_1, \mathbf{v}_1) + C(\Phi, \mathbf{v}_1) = l_1(\mathbf{v}_1) \quad (50)$$

$$\forall \mathbf{v}_2 \in \mathbf{W}_{2a}, \quad a_{2a}(\mathbf{u}_{2a}, \mathbf{v}_2) - C(\Phi, \mathbf{v}_2) = l_{2a}(\mathbf{v}_2) \quad (51)$$

$$\forall \Psi \in M, \quad C(\Psi, \mathbf{u}_1 - \mathbf{u}_{2a}) = 0 \quad (52)$$

where \mathbf{W}_1 , a_1 , l_1 , and C are defined by (12), (14), (16), and (27), respectively, and where:

$$\mathbf{W}_{2a} = \mathbf{H}^1(\Omega_{2a}) \quad (53)$$

$$a_{2a}(\mathbf{u}_{2a}, \mathbf{v}_2) = \int_{\Omega_{2a}} \alpha_2 \boldsymbol{\sigma}(\mathbf{u}_{2a}) : \boldsymbol{\varepsilon}(\mathbf{v}_2) \quad (54)$$

$$l_{2a}(\mathbf{v}_2) = \int_{\Omega_{2a}} \beta_2 \mathbf{f} \cdot \mathbf{v}_2 \quad (55)$$

Under similar hypotheses to those of proposition 3.2, it can be proved that this problem admits a unique solution.

Let us then introduce two global classical (*monomodel*) problems. The first is identical to the reference problem defined in Section 2.1. Its solution is \mathbf{u}^0 . The second is defined in the *altered* domain $\Omega_a = (\Omega_1/\Omega_{12}^f) \cup \Omega_{2a}$, with the same loading than for the previous problem. Its solution is denoted by \mathbf{u}^a . This second problem reads

$$\text{find } \mathbf{u}^a \in \mathbf{W}_a$$

$$\forall \mathbf{v}^a \in \mathbf{W}_a, \quad \int_{\Omega_a} \boldsymbol{\sigma}(\mathbf{u}^a) : \boldsymbol{\varepsilon}(\mathbf{v}^a) = \int_{\Omega_a} \mathbf{f} \cdot \mathbf{v}^a \quad (56)$$

where

$$\mathbf{W}_a = \{\mathbf{v}^a \in \mathbf{H}^1(\Omega_a); \mathbf{v}^a = \mathbf{0} \text{ on } \Gamma_u\} \quad (57)$$

It is well known that under classical regularity hypotheses, this problem admits a unique solution $\mathbf{u}^a \in \mathbf{W}_a$.

For the convergence result given by proposition 4.2, the following lemma (whose proof can be skipped by the noninterested reader) will be useful.

Lemma 4.1 *Under regularity of the domains, under hypotheses (6)–(8), (18), and (19), and for $\kappa_0, \kappa_1 \geq k_0$, with k_0 strictly positive, the semi-norm denoted by ν and defined on \mathbf{W} by*

$$\forall \mathbf{v} \in \mathbf{W}, \quad \nu(\mathbf{v}) = \{a(\mathbf{v}, \mathbf{v}) + C(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)\}^{\frac{1}{2}} \quad (58)$$

where \mathbf{W} , a , and C are defined by (34), (36), and (27), respectively, is a norm on \mathbf{W} , equivalent to the natural one, defined by (35).

Proof. It is quite easy to prove that ν is a norm and that this norm is uniformly bounded by the one defined on \mathbf{W} by (35). It remains, then, to prove that the norm, defined by (35), is uniformly bounded by ν .

Let us distinguish two situations. First, we assume that the irregular α -weight condition (33) is satisfied. Then, by using hypothesis (33), the Korn's inequality, and the fact that the measure of Γ_u (included by hypotheses in the boundary of Ω_1) is strictly positive, we can assert that there exists a constant c (still denoting a generic positive constant independent of \mathbf{v}) such that

$$(\nu(\mathbf{v}))^2 \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_2)\|_{0,\Omega_2}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{0,\Omega_{12}^c}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_1 - \mathbf{v}_2)\|_{0,\Omega_{12}^c}^2 \} \quad (59)$$

Hence

$$(\nu(\mathbf{v}))^2 \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_2)\|_{0,\Omega_2}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{0,\Omega_{12}^c}^2 \} \quad (60)$$

Let us now recall the Young's inequality: For all $\kappa > 0$ and for any two scalars b and d , we have

$$2bd \leq \kappa b^2 + \frac{d^2}{\kappa} \quad (61)$$

By using (60) and the Young's inequality (61), with $b = \|\mathbf{v}_1\|_{0,\Omega_{12}^c}$ and $d = \|\mathbf{v}_2\|_{0,\Omega_{12}^c}$, an appropriate choice of κ (see, e.g., [30] for a similar development), both with a second use of a Korn's inequality, leads us to the following inequality:

$$(\nu(\mathbf{v}))^2 \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\mathbf{v}_2\|_{1,\Omega_2}^2 \} \quad (62)$$

which ends the proof of lemma 4.1 under the hypothesis (33).

Now we assume that the less stringent regular α -weight condition (32) is satisfied. In this case, one of the α_i can be null in some zones of Ω_{12}^c , while the other is equal to 1 in the same zone. To simplify, without reducing the generality of our proof, it can be assumed that α_2 is null in Ω_{12}^c . In this situation, instead of the inequality (59), we have

$$(\nu(\mathbf{v}))^2 \geq c \{ \|\mathbf{v}_1\|_{1,\Omega_1}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_2)\|_{0,\Omega_2/\Omega_{12}^c}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{0,\Omega_{12}^c}^2 + \|\boldsymbol{\varepsilon}(\mathbf{v}_1 - \mathbf{v}_2)\|_{0,\Omega_{12}^c}^2 \} \quad (63)$$

But a use of the Young's inequality, with a second appropriate choice of κ to bound the (tensorial) L^2 -scalar product of $\varepsilon(v_1)$ and $\varepsilon(v_2)$, allows us to derive from (63) an inequality similar to (59). This leads us to the first situation and ends the proof of lemma 4.1.

Now we are able to prove the following relevance result for the Arlequin framework.

Proposition 4.2: A convergence result. *Under the hypotheses of proposition 2.1, the weight parameter indexed family of solutions of the family of problems defined by (50)–(52) and (27), converges strongly, in the following sense:*

if $\beta_2 = 0$ in Ω_{12}^f , then

$$\lim_{\alpha_1/\Omega_{12}^f \rightarrow 1} \|\mathbf{u}_1 - \mathbf{u}^0\|_{1,\Omega} = 0 \quad (64)$$

where \mathbf{u}^0 is the solution of the sound problem, defined by (1)–(5), and if $\beta_1 = 0$ in Ω_{12}^f , then

$$\begin{aligned} \lim_{\alpha_2/\Omega_{12}^f \rightarrow 1} \|\mathbf{u}_1 - \mathbf{u}^a\|_{1,\Omega \setminus \Omega_{12}^f} &= 0 \\ \lim_{\alpha_2/\Omega_{12}^f \rightarrow 1} \|\mathbf{u}_{2a} - \mathbf{u}^a\|_{1,\Omega_{12}^f} &= 0 \end{aligned} \quad (65)$$

where \mathbf{u}^a is the solution of the altered monomodel problem, defined by (56), (57), and (5).

Moreover, the convergence rate is at least of the same order as α_2 in (64) and α_1 in (65).

Proof. We only establish the proof of (65) because the proof of (64) can be carried out similarly (moreover, (65) is the most interesting result in practice). The proof is done in two steps:

1. *A priori estimation.* By taking $v_1 = \mathbf{u}_1$ and $v_2 = \mathbf{u}_{2a}$ in (50)–(55), summing the obtained first two equations and using (27) and the result of lemma 4.1, one can easily check that the family of Arlequin solutions $(\mathbf{u}_1/(\Omega_1/\Omega_{12}^f), \mathbf{u}_{2a})$ is bounded by a strictly positive constant, independent of the weight parameter functions α_i and β_i , $i = 1, 2$.

Now, thanks to this result and to the facts that $\beta_1 = 0$ in Ω_{12}^f and $\mathbf{u}_1 = \mathbf{u}_{2a}$ on $\partial\Omega_{12}^f$, one can easily establish that the family $(\mathbf{u}_1/\Omega_{12}^f)$ is also uniformly bounded, which means that the displacement part of the Arlequin family of solutions is uniformly bounded.

2. *Strong convergence.* Let us now introduce the error-like field

$$e = \begin{cases} \mathbf{u}_{2a} - \mathbf{u}^a & \text{in } \Omega_{2a} \\ \mathbf{u}_1 - \mathbf{u}^a & \text{in } \Omega_1/\Omega_{12}^f \end{cases} \quad (66)$$

where $(\mathbf{u}_1, \mathbf{u}_{2a})$ and \mathbf{u}^a are the solutions of (27) and (50)–(57), respectively. This field e belongs clearly to \mathbf{W}_a .

Let us also define a field e_1 of \mathbf{W}_1 by extending continuously to \mathbf{W}_1 the field of \mathbf{H}^1 (Ω_1/Ω_{12}^f), defined as the restriction to Ω_1/Ω_{12}^f of e , and let us finally define a field e_2 of \mathbf{W}_{2a} as the restriction of e to Ω_{2a} .

By taking $v_1 = e_1$ in (50), $v_2 = e_2$ in (51), and $v^a = e$ in (56), very elementary operations allow us to establish that

$$\int_{\Omega_a} \boldsymbol{\sigma}(e) : \boldsymbol{\varepsilon}(e) = \int_{\Omega_{12}^f} \alpha_1 \{ \boldsymbol{\sigma}(\mathbf{u}_2) : \boldsymbol{\varepsilon}(e) - \boldsymbol{\sigma}(\mathbf{u}_1) : \boldsymbol{\varepsilon}(e_1) \} \quad (67)$$

Now, by using the a priori estimation proved in the first step, the Korn's inequality, and the equation (67), one can easily achieve the proof of the proposition.

By using proposition 4.2 and proposition 3.3, one can check that there is also a convergence of the discrete Arlequin solutions to the solution of the altered monomodel problem when the sizes of the meshes go to zero and when the altered local model is stressed in the sense of proposition 4.2. This result, enlightened by the numerical illustration shown by Fig. 3, is further assessed in the following numerical section.

5. SOME NUMERICAL ILLUSTRATIONS

Let us give other numerical results to enlighten the theoretical investigations.

5.1 Discrete L^2 and H^1 Couplings

The numerical test considered in [3,4] is further commented on here. The mechanical problem consists of evaluating the vertical displacement field u in an upright elastic bar, occupying the closure of the domain $\Omega =]0, 3[$, of constant section and of Young's modulus and mass density E and ρ , respectively, clamped at both ends and loaded by its own weight (see Fig. 4(left)).

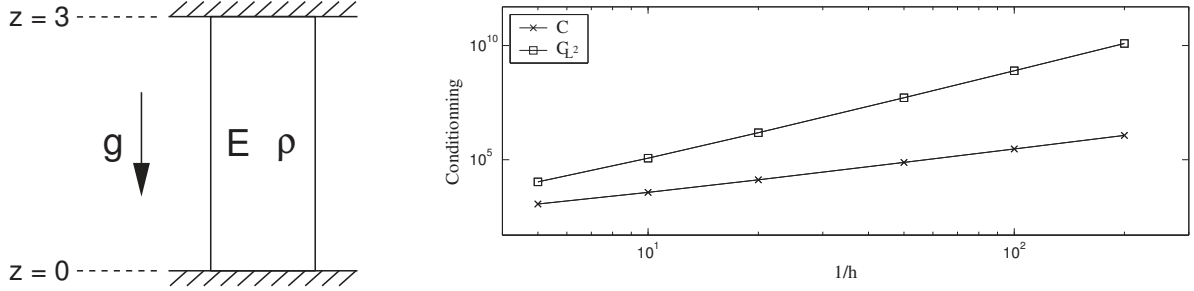


FIGURE 4. Clamped vertical bar and comparison of the conditioning of the discrete Arlequin systems associated with energy and L^2 couplings

The discrete finite element Arlequin model for this problem consists of the superposition of two equally fine meshes set in $]0, 2[$ and $]1, 3[$, with step denoted by h . The weight functions α_i and β_i , $i = 1, 2$, are taken to be constant and equal to $1/2$ in the gluing zone Ω_{12}^c , coinciding here with the overlap Ω_{12} . Continuous 1-D linear elements are used to approximate the displacement and Lagrange multiplier fields. In Fig. 4(right), we compare the influence of the mesh size h on the conditioning of the linear Arlequin systems associated with the H^1 and L^2 scalar products couplings. Clearly the C coupling operator leads to conditioning numbers varying as $1/h^2$ (which is usual for finite element matrices), whereas the L^2 scalar product, denoted by C_{L^2} and approximating in the discrete finite element model the dual C_d coupling operator, leads to worse conditioning numbers, varying as $1/h^4$. The different behaviors of the Lagrange multiplier field, denoted λ_h in both cases, are represented in Fig. 5, for the L^2 coupling in the left and for the H^1 coupling in the right. One can observe that the latter leads to a smooth field, while the former gives a singular Lagrange multiplier: We observed that when h goes to zero, this multiplier tends to be localized at the boundary of the coupling zone, with a numerical value on the two points of the boundary varying as $1/h$. Actually, up to a homogenization factor

of order $1/h$, it seems to converge to a surface Lagrange multiplier. This simple result enlightens the comment given at the end of Section 2.2.1, where we stressed the irrelevance of a continuous L^2 scalar product coupling operator: The discrete Lagrange multiplier fields, obtained with this coupling operator, for the example considered here are singular. It cannot be an approximation of an L^2 field.

5.2 A Local Loss of Stability Test

We consider the example of the elastic bar depicted by Fig. 4, with the same data. This bar is now calculated by using two different but compatible meshes for $\Omega_1 =]0, 2[$ and $\Omega_2 =]1, 3[$. Coarse and fine linear finite elements are used for the respective approximation of the displacement fields u_1 and u_2 . The restriction of the coarse finite element space to the overlap $\Omega_{12} =]1, 2[$ ($= \Omega_{12}^c$, in the considered example) is used for the approximation of the mediator space. The weight functions are regular, with $\alpha_1 = \beta_1 = 1$, in Ω_1/Ω_{12} and varying as Hermite cubic functions in Ω_{12} . The results (analytical and Arlequin displacements) are plotted in Fig. 6, where we have varied the fine mesh size, while keeping the coarse one constant. Only the overlap zone is shown. One can make the following observations:

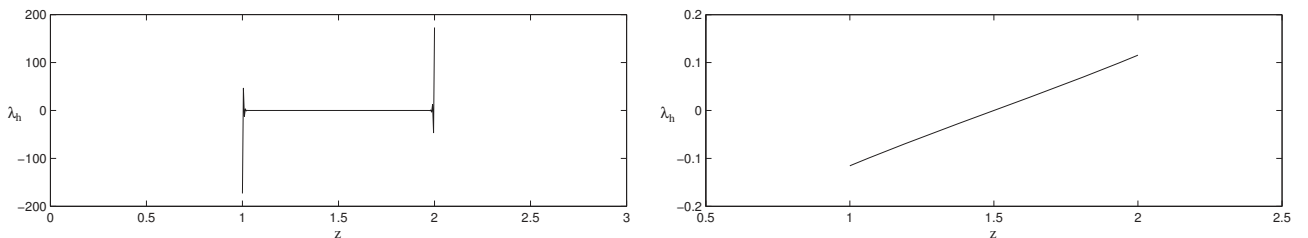


FIGURE 5. L^2 and H^1 coupling: comparison of Lagrange multiplier fields for the 1-D bar problem, with $h = 0.05$

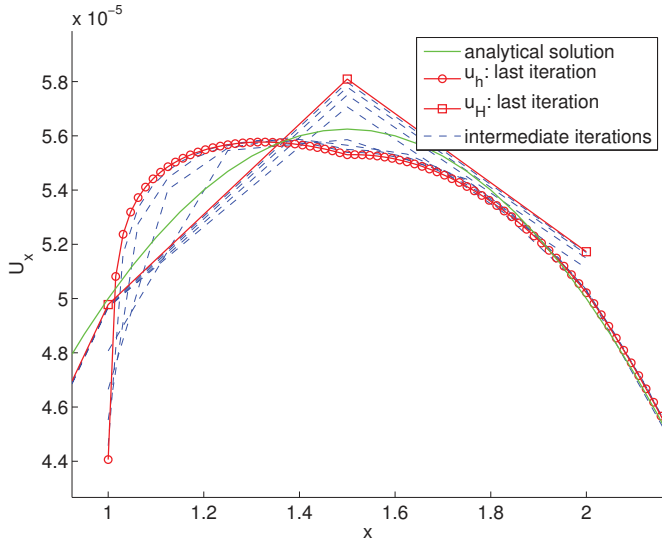


FIGURE 6. A local loss of stability due to a limit local loss of coercivity (computed by O. Jamond)

- The coarse and fine displacement fields are not equal to each other in the overlap. This is due to the fact that their equality is only enforced by means of a projection on the coarse *mediator* space.
- The converged fine part of the Arlequin solution is very accurate on the right part of the overlap, where the fine weight functions are near 1.
- A kind of instability of the solution u_2 in the vicinity of the left boundary of Ω_2 , where the fine internal weight parameter α_2 comes to zero, clarifies the comment concerning the local loss of *coercivity*, given at the end of Section 3.2.3.

These results seem to indicate that when choosing regular weight functions, the coarse, fine, and *mediator* spaces should be such that the Arlequin problem stays stable and the coarse and fine solutions are sufficiently close in the zones where the fine internal weight function α_2 becomes near zero. In practice, this can be achieved in different ways (e.g., by using a kind of progressive refinement of the *mediator* space).

By taking in the considered test M_{ht} equal to the restriction of the finer finite element space to Ω_{12} , the obtained numerical results do not show any instability and the Arlequin solution coincides (in the

overlap) with the coarse monomodel finite element approximation. Sometimes this strong enforcement of the equality between the two fields has been abusively qualified as a locking phenomenon.

5.3 A Holed Domain: Comparison of the Numerical and Analytical Results

We consider here the test described by Fig. 7. It consists of a holed elastic medium calculated as a 2-D problem under plane strain hypotheses. The medium is assumed to be submitted to a uniform traction f on its upper and lower edges and free on the two other edges (see Fig. 7). The data are $E = 300$ GPa, $\nu = 0,3$, $L = 40$ m, $R = 1,3$ m, and $f = 2$ MPa. The Analytical value of $\sigma_{\theta\theta}$ along the perimeter of the hole is given by, (e.g., [31])

$$\sigma_{\theta\theta} = f(1 - 2 \cos 2\theta) \quad (68)$$

The problem is calculated in the Arlequin framework by superimposed a local holed elastic model to a global elastic sound model with a constant α_2 associated to the holed local model equal to 0.999. The local holed finite element model is fine to capture the stress concentration, and the sound global model is coarse. The deformed meshes with the iso-principal stresses are represented in the left part of Fig. 8. The numerical $\sigma_{\theta\theta}$ solution along the hole perimeter is compared to the analytical one in the right part of Fig. 8. They are quite similar.

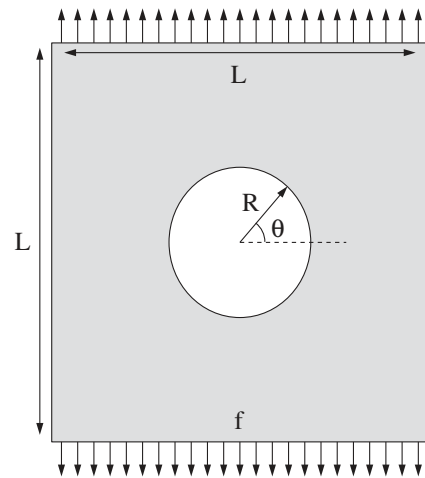


FIGURE 7. A holed domain

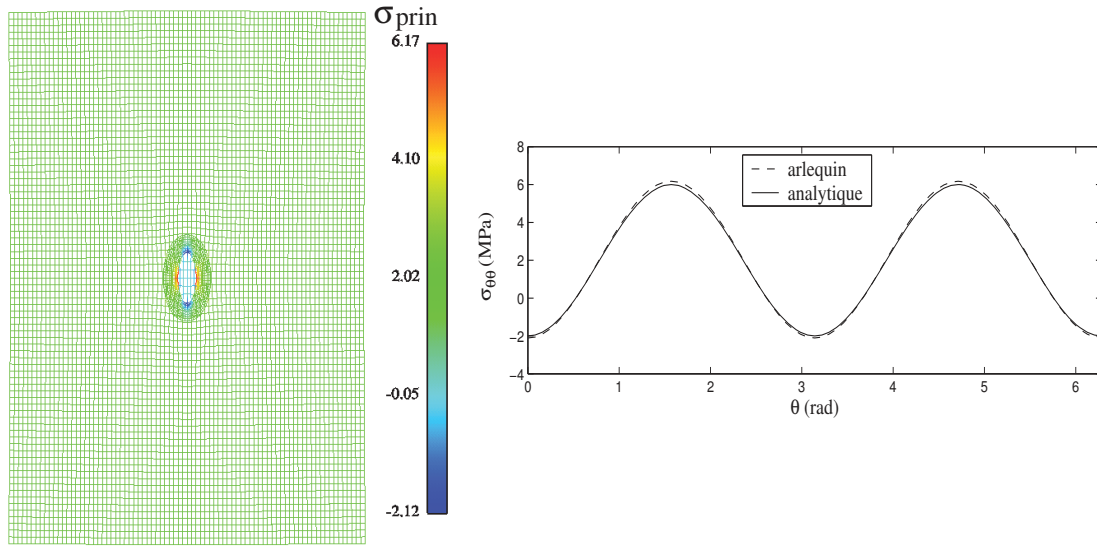


FIGURE 8. A holed local model superposed to a sound model: deformed Arlequin model, iso-major principal stresses, and comparison of the numerical solution to the analytical one

5.4 A Convergence Result for a Quantity of Interest

This last numerical result enlightens the theoretical one, given by proposition 4.2. We consider a slanted, cracked plate under traction loads, as shown by Fig. 9. The data are $E = 200$ GPa, $\nu = 0,3$, $f = 100$ MPa, $a = 1$ mm, and $\beta = 37$. Numerical tests are carried out with α_2 (denoted by α_{crack} in Fig. 11), taking values that go progressively to 1. The global meshed model with a zoom around the crack is given by Fig. 1.

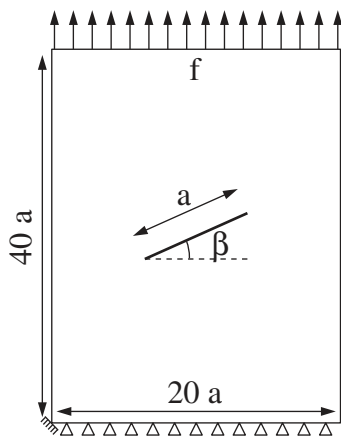


FIGURE 9. A cracked plate is superimposed locally on a sound global plate

The iso-major stresses on the deformed zoom and on a zoom of the zoom around one crack tip are depicted in Fig. 10.

The Arlequin numerical energy restitution rates G are compared, in Fig. 11, to the numerical energy restitution rate obtained by a globally refined monomodel, and to the analytical solution (e.g., [32]) are given by

$$G = \frac{f^2}{E} \pi a \cos^2 \beta \quad (69)$$

The convergence (up to small errors introduced by the finite element approximations) of the Arlequin energy restitution rates to the analytical restitution rate when α_{crack} goes to 1 is clear.

6. CONCLUSIONS

We have herein given an advanced presentation of the multimodel and multiscale Arlequin framework. The definition of reliable sets of Arlequin's parameters has been given precisely by further theoretical analyses. Our most noticeable conclusions are the following:

1. The boundary of the gluing zone has to include the boundary of the overlap.
2. We need a regularity of the free zone to ensure a uniform *Inf-Sup* condition.

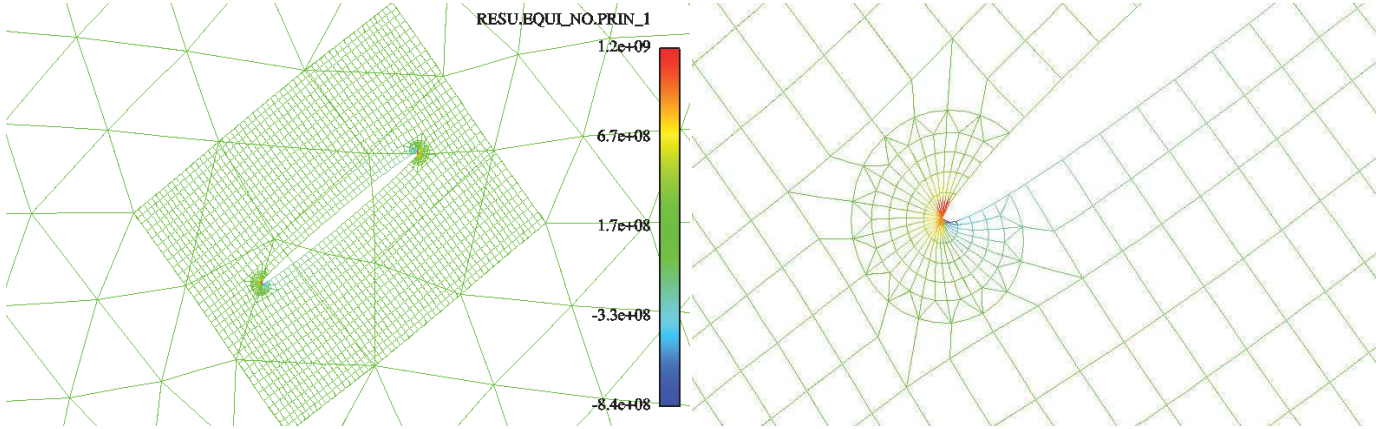


FIGURE 10. A cracked plate is superimposed locally on a sound global plate

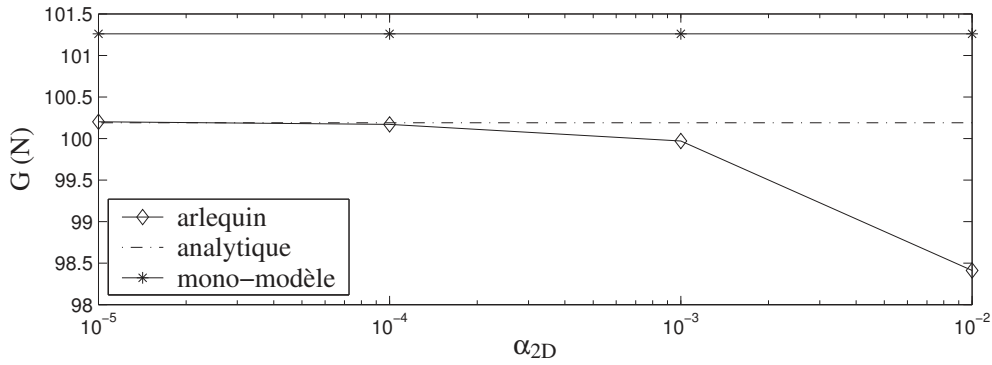


FIGURE 11. A convergence result

3. The alteration has to be strictly embedded in the free zone.
4. The solutions related to the superimposed models have to be quite similar in the gluing zone.
5. The finite element L^2 coupling is an approximation of the continuous duality coupling that leads to ill-conditioned Arlequin problems (unless scaled appropriately).
6. The weight parameter functions have to satisfy the conditions (18) and (19), and it is possible to use regular internal weight parameter functions verifying the condition (32), while ensuring the stability of the continuous and discrete Arlequin problems. However, the stability of the latter may become mesh-dependent when a fine mesh is locally superposed to a coarse

mesh and when the *mediator* space is chosen equal to the restriction to the gluing zone of the coarse finite element space.

7. It seems important (for consistency and for the relevance of results) to take the internal weight parameters α_i to be constant in the free part Ω_{12}^f of the overlapping zone.
8. To let the alteration of a global sound model be effective, the weight parameter associated with the locally superposed and altered model has to be stressed, by taking it (constant) near 1 in the free zone (as shown by Fig. 2).

Our theoretical analyses have been enlightened by numerical tests, some of which had already been published but were shown herein to assess our theoretical investigations.

From another side, it is believed that the Arlequin framework can be used in many other fields of in-

terst. For instance, it can be used to capture boundary layers (e.g., in fluid dynamics) or to couple locally a numerical solution to an experimental one. It can also be extended to mechanical multifield mixed problems of De Veubeke-Hu-Washizu type [33–35]. Notice that for these mixed problems, local strains and stresses can then be part of the coupled fields in the gluing zones Ω_{12}^c .

Finally, the first results concerning the coupling of atomistic and continuum models in the Arlequin framework are promising (see, e.g., [13,14,16]). We are actively pursuing this work by testing other accommodation operators than the interpolation one and by approximating the atomistic energy in the gluing zone (or a part of it) by the consistent continuum energy (when available [36]), evaluated at the accommodated atomistic fields.

REFERENCES

1. Dhia, H. B., Multiscale mechanical problems: The Arlequin method. *Comptes Rendus Acad. Sci. Sér. IIB*. **326**:899–904, 1998.
2. Dhia, H. B., Numerical modeling of multiscale problems: The Arlequin method. *CD Proceedings of ECCM'99: First European Conference on Computational Mechanics, Muenchen, August 31–September 3*, Wunderlich, W., Ed., 1999.
3. Dhia, H. B., and Rateau G., Application of the Arlequin method to some structures with defects. *Rev. Europ. Éléments Finis*. **11**:291–304, 2002.
4. Dhia, H. B., and Rateau G., The Arlequin method as a flexible engineering design tool. *Int. J. Numer. Methods Eng.* **62**:1442–1462, 2005.
5. Dhia, H. B., and Zarroug M., Contact in the Arlequin framework. *Contact mechanics*. Martins, J. A. C., and Monteiro Marques, M. D. P., Eds., Kluwer, New York, 401–410, 2001.
6. Rateau, G., Méthode Arlequin pour les Problèmes Mécaniques Multi-échelles. Application à des Problèmes de Jonction et de Fissuration de Structures Élancées. PhD Thesis, Ecole Centrale Paris, 2003.
7. Dhia, H. B., and Zammali, C., Level-Sets and Arlequin framework for dynamic contact problems. *Revue Euro. Éléments Finis*. **13**:403–414, 2004.
8. Dhia, H. B., and Zammali, C., Mutiscale analysis of impacted structures. *CD Proceedings of IASS/IACM 2005: 5th International Conference on Computation of Shell and Spatial Structures, 1-4 June*, Salzburg, Austria, 2005.
9. Dhia, H. B., Global-local approaches: The Arlequin framework. *Euro. J. Comput. Mech.* **15**:67–80, 2006.
10. Guidault, P.-A., and Belytschko, T., On the L^2 and the H^1 couplings for an overlapping domain decomposition method using Lagrange multipliers. *Int. J. Numer. Methods Eng.* **70**:322–350, 2007.
11. Xiao, S. P., and Belytschko, T., A bridging domain method for coupling continua with molecular dynamics. *Comput. Methods Appl. Mech. Eng.* **193**:1645–1669, 2004.
12. Fish, J., Nuggehally, M. A., Shephard, M. S., Picu, C. R., Badia, S., Parks, M. L., and Gunzburger, M., Concurrent AtC coupling based on blend of continuum stress and the atomistic force. *Comput. Methods Appl. Mech. Eng.* **196**:4548–4560, 2007.
13. Dhia, H. B., and Elkhodja, N., Coupling of atomistic and continuum models in the Arlequin framework. *Proceedings of the 8eme Congrès de Mécanique, El Jadida, Maroc, April 17–20*, 133–135, 2007.
14. Bauman, P. T., Dhia, H. B., Elkhodja, N., Oden, J. T., and Prudhomme, S., On the application of the Arlequin method to the coupling of particle and continuum models. *Comput. Mech.* 2008.
15. Badia, S., Bochev, P., Lehoucq, R., Parks, M. L., Fish, J., Nuggehally, M. A., and Gunzburger, M., A force-based blending model for atomistic-to-continuum coupling. *Int. J. Multi-scale Comput. Eng.* **5**:387–406, 2007.
16. Prudhomme, S., Dhia, H. B., Bauman, P. T., Elkhodja, N., and Oden, J. T., Computational analysis of modeling error for the coupling of particle and continuum models by the Arlequin method. *Comput. Methods Appl. Mech. Eng.* in press.
17. Badia, S., Parks, M., Bochev, P., Gunzburger, M., and Lehoucq, R., On atomistic-to-continuum (AtC) coupling by blending. *Multi-scale Model. Simul.* **7**:362–380, 2008.

18. Dhia, H. B., and Rateau G., Mathematical analysis of the mixed Arlequin method. *Comptes Rendus Acad. Sci. Paris Série I.* **332**:649–654, 2001.
19. Gurtin, M. E., *An Introduction to Continuum Mechanics*. Academic, New York, 1981.
20. Rudin, W., *Functional Analysis*. McGraw Hill, New York, 1973.
21. Babuška, I., Error bounds for finite element method. *Numer. Math.* **16**:322–333, 1971.
22. Brezzi, F., On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *R.A.I.R.O. Anal. Numér.* **8**:129–151, 1974.
23. Adams, R., *Sobolev Spaces*. Academic, New York, 1976.
24. Grisvard, P., *Elliptic Problems in Nonsmooth Domains*. London, Pitman, 1985.
25. Ciarlet, P. G., *The Finite Element Method for Elliptic Problems*. North-Holland, 1978.
26. Zienkiewicz, O. C., and Taylor, R. L. G., *The Finite Element Method*. 2 vols. McGraw-Hill, New York, 1999–2001.
27. Dauge, M., *Elliptic Boundary Value Problems on Corner Domains*. Springer, Paris, 1988.
28. Widlund, O. B., An extension theorem for finite element spaces with three applications. *Proceedings of a GAMM Seminar on Numerical Techniques in Continuum Mechanics, Kiel, Germany, January 17–19, 1986*, Hackbusch, W., and Witsch, K., Eds., Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, 110–122, 1987.
29. Bjorstad, P. E., and Widlund, O. B., Iterative methods for the solution of elliptic problems in regions partitioned into substructures. *SIAM J. Numer. Anal.* **23**:1097–1120, 1986.
30. Dhia, H. B., Mathematical analysis of non-linear thin plate problems of Mindlin-Naghdi-Reissner type: Existence of solutions under optimal hypotheses. *C. R. Acad. Sci. Paris, Ser. I.* **320**:1545–1552, 1995.
31. Muskhelishvili, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Leiden, 1954.
32. Murakami, Y., *Stress Intensity Factors Handbook*. Pergamon, Oxford, 1987.
33. de Veubeke, B. M. F., Diffusion des inconnues hyperstatiques dans les voilures à longeron couplés. *Bull. Serv. Tech. Aéronaut.* **24**: 1951.
34. Hu, H.-C., On some variational methods on the theory of elasticity and the theory of plasticity. *Scientia Sinica.* **4**:33–54, 1955.
35. Washizu, K., On the variational principles of elasticity and plasticity. Report 25–18. Massachusetts Institute of Technology, Boston, 1955.
36. Blanc, X., Le Bris, C., and Lions, P.-L., From molecular models to continuum mechanics. *Arch. Rat. Mech. Anal.* **164**:341–381, 2002.