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Abstract

This paper develops a method of stability analysis of linear time-delay systems with commensurate delays and delay-dependent coefficients. The method is based on a D-decomposition formulation that consists of identifying the critical pairs of delay and frequency, and determining the corresponding crossing directions. The process of identifying the critical pairs consists of a magnitude condition and a phase condition. The magnitude condition utilizes the Orlando's formula, and generates frequency curves within the delay interval of interest. Such frequency curves correspond to the the delay-frequency pairs such that the decomposition equation has at least one solution on the unit circle. The delay interval of interest is divided into continuous frequency curve intervals (CFCIs). Under some non-degeneracy assumptions, the number of frequency curves remains constant within each CFCI, and the associated decomposition equation has one and only

one solution on the unit circle at any point on a frequency curve. By traversing through the frequency curves, all the crossing points can be identified. The crossing direction is related to the sign of the lowest-order nonzero derivative of the phase angle with respect to the delay, which is a generalization of the existing literature even for the case with single delay. This conclusion allows one to determine the crossing direction by examining the phase angle vs delay diagram. An example is presented to illustrate how a stability analysis can be conducted if some non-degeneracy assumptions are violated.

Keywords: Stability, differential-difference equations, time-delay systems, delay-dependent coefficients, commensurate delays

1 Introduction

This paper develops a method of analyzing stability of linear time-delay systems described by differential-difference equations with commensurate delays and delay-dependent coefficients. The corresponding characteristic equation has the following general form

$$D(\lambda,\tau)=\sum_{k=0}^{M}P_k(\lambda,\tau)e^{-k\lambda\tau}=0, \tag{1}$$

where $P_k(\lambda,\tau)$ for each given k is continuous with respect to the delay τ and is a polynomial of the Laplace transform variable λ with real coefficients for any given $\tau\in[\tau^l,\tau^u]$. Such a system may arise from various scientific disciplines, such as population models with age structure [3], the stellar dynamos [40], and hematopoiesis dynamics [9]. The delay parameter in the coefficients may also arise from approximating derivative action by finite-difference in feedback control [23, 24, 36–38] (especially, see [29] for an example of commensurate delay approximation of derivative in feedback control), linearization of nonlinear models about equilibria that depend on the delay, and the problem of requiring guaranteed exponential convergent rate as will be shown in Section 2.

In spite of their practical importance, most existing approaches in the literature only consider systems with coefficients of the characteristic quasipolynominal independent of the delay. A notable exception is due to Berreta and Kuang [3], where an effective method was developed to determine stability of systems with a single delay and delay-dependent coefficients. In [20], this was generalized to a wider-class of systems, and the structure of the set of critical delay-frequency pairs are clearly described.

In [21], which may be considered as a preliminary version of this paper, the results of [20] are extended to the case of commensurate delays. In addition, the crossing direction analysis has been extended from the case that uses only the first-order derivative to the case where higher-order derivatives are needed, which is a generalization even for the single delay case. The analysis in [21] was based on some assumptions that turn out to be either overly restrictive or unnecessary, and some proofs are either omitted

or only sketched. No procedure is given for the calculations of some quantities used in the stability analysis.

In this paper, a more careful analysis allows us to show the implications of the four nondegeneracy assumptions. As a result, some assumptions in [21] are either relaxed or eliminated. The resulting process for stability analysis is simpler and more parallel to the single delay case. Complete proofs are provided and made more rigorous, and procedure of calculating the quantities needed for stability analysis are provided. Specifically, Assumptions II, III and IV in [21] are relaxed to Assumptions II, IV and III in this paper. In the analytical crossing direction analysis, Assumption IIIa is introduced, and its relationship with Assumption III, which parallels the single delay case, is presented. A new theorem (Theorem 1) has been introduced, which made Assumption V in [21] unnecessary. This also made the stability analysis process described Section 3 of this paper simpler and more parallel to the single delay case given in [20]. An important basis of establishing Theorem 1 is understanding the cases where no unit circle solution exists even though the Orlando magnitude equation is satisfied (see the discussions after Lemma 2 and Examples 1, 2 and 3). In the crossing direction analysis described in Section C of [21], only a sketch for the proof of the main result, Theorem 1, was provided, and the proof for Proposition 3 (which supports Theorem 1) was omitted. Furthermore, no procedure for the calculation of the quantities in Theorem 1 was provided. In this paper, the presentation of crossing direction analysis has been reorganized, the proofs are provided and made more rigorous. Four appendices have been introduced to provide background, cover technical details, and provide procedure of calculting quantifies necessary for crossing direction analysis. Additional observations are made to facilitate the analysis, including the possibility of making judgement of crossing direction by observing the crossing of $r2\pi$ lines by the phase vs delay diagram (see Corollary 2), and invariance of two quantities used in the analysis (see Proposition 1).

In the following, we will provide a more extensive review of the background literature and highlight the new results in this paper and their significance.

1.1 *D*-decomposition

D-decomposition method [10] is a very effective method in the stability analysis of time-delay systems. The method is based on identifying the set of parameters, which we call stability crossing set [14], for which there is at least one characteristic root on the imaginary axis. In most practical cases, the characteristic roots depend on the parameters continuously. As a result, the stability crossing set divides the parameter space into regions with fixed number of characteristic roots on the right half-plane (RHP). By determining the direction the characteristic roots cross the imaginary axis, the number the RHP characteristic roots in each such region can be determined, especially, the parameter regions where there is no RHP characteristic root (thus the system is stable) can be identified.

When the parameter concerned is the delay, the D-decomposition method is also known as the τ-decomposition method [27]. Due to periodic nature of the exponential function with imaginary exponent, significant complication may arise as compared with D-decomposition problem where no delay parameter is involved.

1.2 System with single delay and delay-dependent coefficients

In [3], the crossing direction in a neighborhood of a crossing pair was studied. In [20], an assumption in [3] was relaxed, and a method of searching all critical pairs was developed. In the following, we will outline the main idea of the method developed in [20] using a slightly different notation and terminology in order to make its relationship with the current paper more clear. For a system with characteristic equation

$$D_{(s)}(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{2}$$

using the same principle as used in [39] for the single delay case, the critical pairs (ω, τ) may be identified by using a magnitude condition

$$F_{(s)}(\omega, \tau) = P(j\omega, \tau)P(-j\omega, \tau) - Q(j\omega, \tau)Q(-j\omega, \tau) = 0 \tag{3}$$

and a phase condition

$$\angle\theta_{(s)}(\omega, \tau) = \angle P(j\omega, \tau) - \angle Q(j\omega, \tau) + \omega\tau + \pi = 2r\pi. \tag{4}$$

Under nondegeneracy assumptions, the interval of interest $\mathcal{T}$ may be divided into subintervals $\mathcal{T}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}]$, which will be called Continuous Frequency Curve Intervals (CFCIs) in this paper. The dividing points $\tau^{(i)}$, $i = 1, 2, \ldots, K-1$ satisfy

$$\partial_\omega F_{(s)}(\omega, \tau) = 0 \tag{5}$$

for some ω. The number of real solutions of (3) for ω remains constant within the interior of each subinterval $\mathcal{T}^{(i)}$, and these solutions are continuous functions $\omega_k^{(i)}(\tau)$, $k = 1, 2, \ldots, m^{(i)}$ of τ. By traversing through these curves, all the critical pairs can be identified by using the phase condition (4).

1.3 System with multiple delays

Many practical systems have multiple delays. Such systems have a characteristic equation of the form

$$D(\lambda) = \sum_{k=0}^{M} P_k(\lambda)e^{-\tau_{(k)}\lambda} = 0, \tag{6}$$

where $P_k(\lambda)$ are polynomials. Earlier studies consider systems with two delays. See [16, 32, 34] for the special case of $P_k(\lambda)$ =constant, [35, 42] for two studies of the general case, [15] for a complete description of the problem structure and a procedure of stability analysis. When the coefficient polynomials P_k also depend on the delays, [1] extended the method in [15] to an interesting special class of such systems. Unfortunately, a thorough analysis of such systems with more than two delays becomes much more complicated [5, 11, 14] .

1.4 System with commensurate delays

In some systems, the delays $\tau_{(k)}$ in (6) are constrained to be commensurate, i.e., $\tau_{(k)} = k\tau$ for some $\tau > 0$. Such systems arise naturally in practice, such as from using more than two terms to more accurately approximate derivatives in feedback control [38] and from multivariable differential-difference equations with a single delay as will be shown in Section 2. The stability analysis for such systems with delay-independent coefficients has been carried out in, for example, [6, 7, 18, 19, 39].

This paper considers the stability analysis of systems with commensurate delays and delay-dependent coefficients. Similar to the single delay case discussed in [20], the conditions for the critical pairs consist of a magnitude condition and a phase condition. However, the magnitude condition is defined implicitly by requiring the following equation

$$\sum_{k=0}^{M} P_k(j\omega, \tau)x^k = 0 \tag{7}$$

to have at least one solution on the unit circle. The phase condition is expressed in terms of the phase angles of such unit circle solutions.

A method to write the magnitude condition in an explicit form is to apply the Orlando's formula [22, 33], which is presented in Appendix A. However, a number of complications arise: 1) There may be cases that such explicit form of magnitude condition is satisfied, but the equation (7) does not have any solution on the unit circle; 2) The equation (7) may have more than one solutions on the unit circle. To circumvent these complications, some assumptions are made regarding the invariance of unit circle solutions of (7) in [21], which may be considered as a preliminary verison of this paper. In this paper, it was shown that the nondegeneracy assumptions imply that the explicit form of magnitude condition guarantees that the equation (7) has one and only one solution on the unit circle when the delay is restricted to the interior of any CFCI. This significantly simplified the structure of the problem. As a result, it is possible to make the solution process more parallel to the single delay case.

In addition, this paper also generalized the analytical crossing condition from the case that can be determined by the first-order derivative to the one that requires higher-order derivatives.

Arriving at these results depends on an analysis of the derivatives of polynomial roots with respect to parameters which may be constrained by another equation. Such analysis is more subtle, and requires some additional tools as compared with [20]. For example, when a multiple solutions of polynomial is concerned, a puiseux series [25, 26] is needed for such analysis, and a higher-order derivatives of one variable with respect to one or more other variables defined by implicit function relations are needed.

1.5 Outline of the paper

Section 2 sets up the problem, defines the magnitude and phase conditions, introduces the nondegeneracy assumptions, and discusses the implications of these assumptions. Section 3 discusses the stability analysis, including the description of continuous frequency curves and continuous frequency curve intervals, the determination of critical

pairs, the crossing directions, culminating in the determination of stable delay intervals. Section 4 presents an analytical method to determine the crossing direction of critial pairs that generalizes the existing results to one that may require higher-order derivatives, and relate the crossing direction of the characteristic roots at the imaginary axis to that of the phase angle as a function of delay at some designated horizontal lines. Some invariant results are also presented. Section 5 presents some illustrative examples, including one that illustrates the stability analysis of a system that violates some nondegeneracy assumptions. Section 6 presents some conclusions of this paper and potential further research. Four appendices are included to provide supplementary materials. Appendix A presents the Orlando's formula. Appendix B discusses the derivatives of one variable with respect to one or more other variables defined by implicit functions. Appendix C discusses the derivatives of phase angle as a function of delay. Appendix D presents the proof of one of the main theorems (Theorem 2).

1.6 Notation

For a complex number c, $\mathcal{R}(c)$, $\mathcal{I}(c)$ and $\bar{c}$ denote its real part, imaginary part and complex conjugate, respectively. The unit circle in the complex plane is denoted by $\mathbb{S}$. $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}_+$ for positive reals, and $\bar{\mathbb{R}}_+$ for nonnegative reals. For a function defined in a closed interval, say $[a, b]$, its derivatives at a and b are understood as the right and left derivative, respectively. We sometimes use ∂ with a subscript to denote partial derivatives. For instance, $\partial_\lambda D(\lambda, \tau) := \frac{\partial D(\lambda,\tau)}{\partial \lambda}$. For a function with two variables, we may write one variable as a subscript to indicate that we consider this variable as a fixed parameter. For example, if $D(\lambda, \tau)$ is continuous function of τ and polynomial of λ, then $D_\tau(\lambda) = D(\lambda, \tau)$. With this notation, we can say $D_\tau(\lambda)$ is a polynomial. Similarly, for a function of three variables such as $\Phi(\omega, \tau, x)$, then $\Phi_{\omega\tau}(x) = \Phi(\omega, \tau, x)$, but we consider $\Phi_{\omega\tau}(x)$ as a function of x, and ω and τ as fixed parameters. For a function of single variable $f(x)$, its derivative is denoted by $f'(x)$. For a polynomial $P(\lambda)$, its degree is denoted as $\deg(P)$[1].

2 Systems with commensurate delays

Consider a system represented by a multivariable differential-difference equation with the characteristic equation given in the equation (1), where each $P_k(\lambda, \tau)$ is continuous with respect to the delay τ and is a polynomial of the Laplace transform variable λ with real coefficients for any given $\tau \in \mathcal{T}$, and $\mathcal{T} = [\tau^l, \tau^u]$ is the delay interval of interest. For a $\tau \in \mathcal{T}$, let $N^u(\tau)$ be the number of characteristic roots of (1) on the strict RHP for the given τ. We assume $N^u(\tau^l)$ is known (well-known methods such as the one given in [39] may be used to determine this). Our objective is to find the set of τ contained in $\mathcal{T}$ for which the system with characteristic equation (1) is asymptotically stable.

The characteristic equation given in (1) has commensurate delays and delay-dependent coefficients. As indicated in the introduction, this form of characteristic equation appears frequently in practical systems. As a specific example, consider a

[1] This was denoted as "ord(P)" representing the order of the polynomial in [20, 21]. We now use the term "degree" instead to avoid potential ambiguity.

system represented by the following multi-variable differential-difference equation,

$$\dot{x}(t) = A_1(\tau)x(t) + A_2(\tau)x(t-\tau),$$

where A_1 and A_2 are real matrices that depend on τ. It is easily verified that its characteristic equation is indeed of the form (1) with M equal to the dimension of x. Note that in this case commensurate delays may appear in the corresponding characteristic equation even though there is only one delay in the original differential-difference equation. In addition, characteristic equations with delay-dependent coefficients may arise from the α-stability analysis of systems with delay-free coefficients. Consider a system with the characteristic equation [6], [18]:

$$\lambda + e^{-\tau\lambda} + e^{-2\tau\lambda} = 0. \tag{8}$$

The system is said to be α-stable if the real parts of all the roots of (8) are less than $-\alpha$. Replacing λ with $\lambda - \alpha$ in (8), we obtain

$$\lambda - \alpha + e^{\alpha\tau}e^{-\tau\lambda} + e^{2\alpha\tau}e^{-2\tau\lambda} = 0. \tag{9}$$

It is easy to see that the α-stability of the system with characteristic equation (8) is equivalent to the asymptotic stability of the system with characteristic equation (9), which has delay-dependent coefficients, even though the coefficients of (8) are independent of the delay τ.

It is instrumental to define

$$\Phi(\omega, \tau, x) = \sum_{k=0}^{M} P_k(j\omega, \tau)x^k, \tag{10}$$

and

$$\Psi(\omega, \tau, x) = \sum_{k=0}^{M} P_k(-j\omega, \tau)x^{M-k}. \tag{11}$$

According to the definition,

$$D(j\omega, \tau) = \Phi(\omega, \tau, e^{-j\omega\tau}). \tag{12}$$

Then the following lemma is obvious.

Lemma 1. *The characteristic equation (1) has a solution for λ on the imaginary axis, i.e.,*

$$D_\tau(j\omega) = 0 \tag{13}$$

for some $\omega \in \mathbb{R}$ if and only if

$$\Phi_{\omega\tau}(x) = 0, \tag{14}$$

or equivalently,

$$\Psi_{\omega\tau}(x) = 0, \tag{15}$$

has a solution on the unit circle $x \in \mathbb{S}$ *such that*

$$\angle x = -\omega\tau + r2\pi \tag{16}$$

for some integer r*.*

Proof. The equations (14) with $x \in \mathbb{S}$ and (16) are equivalent to (13) in view of (12). Therefore, it is sufficient to show that (14) and (15) are equivalent for $x \in \mathbb{S}$. Take complex conjugate of (14), multiply the resulting equation by x^M and observe that $\bar{x} = x^{-1}$ when $x \in \mathbb{S}$, we obtain (15). This process is reversible. □

A crucial step in carrying out the τ-decomposition process is to identify all the pairs $(\omega, \tau) \in \mathbb{R}_+ \times \mathcal{T}$ such that (13) is satisfied. Parallel to the single delay case discussed in [20], Lemma 1 decomposes the requirement (13) into two parts: the *magnitude condition* (i.e., (14) is satisfied for some $x \in \mathbb{S}$) and the *phase condition* (16). Because of its role in the decomposition to the magnitude condition and the phase condition, the equation (14) will be referred to as the *decomposition equation*. Similarly, the equation (15) will be referred to as the *alternate decomposition equation*.

While the magnitude condition in the single delay case is very simple, the commensurate delay counterpart discussed here is defined implicitly. Fortunately, an explicit condition is still possible by utilizing the Orlando's formula, which is restated in Appendix A. Define the Schur-Cohn-Fujiwara matrix [6, 7, 18, 19],

$$\Theta(\omega, \tau) = \Psi^H(\omega, \tau, S)\Psi(\omega, \tau, S) - \Phi^H(\omega, \tau, S)\Phi(\omega, \tau, S), \tag{17}$$

where a supercript H represents conjugate transpose of the matrix, and S is the $M \times M$ shift matrix defined in (A4) in Appendix A. Define

$$F(\omega, \tau) = -\det \Theta(\omega, \tau). \tag{18}$$

Then the magnitude condition can be expressed in terms of F in view of the following lemma.

Lemma 2. *If* ω *and* τ *are such that*

$$P_M(j\omega, \tau) \neq 0, \tag{19}$$

then

$$F(\omega, \tau) = -|P_M(j\omega, \tau)|^{2M} \prod_{i=1}^{M} \prod_{k=1}^{M} (1 - x_{\Phi i}\bar{x}_{\Phi k}), \tag{20}$$

where $x_{\phi i}, i = 1, 2, \ldots, M$ *are the* M *solutions of the decomposition equation (14). If* ω *and* τ *satisfy*

$$P_0(j\omega, \tau) \neq 0, \tag{21}$$

then

$$F(\omega,\tau) = (-1)^{M+1} \left|P_0(j\omega,\tau)\right|^{2M} \prod_{i=1}^{M}\prod_{k=1}^{M}(1 - x_{\Psi i}\bar{x}_{\Psi k}), \tag{22}$$

where $x_{\Psi i}, i = 1, 2, \ldots, M$ *are the* M *solutions of the alternate decomposition equation (15).*

Proof. The expression (20) is obtained by applying the Orlando's formula given in Appendix A to the polynomial $\Phi_{\omega\tau}(x)$. The expression (22) is obtained by applying the Orlando's formula to the polynomial $\Psi_{\omega\tau}(x)$. □

It is immediately clear from Lemma 2 that

$$F(\omega,\tau) = 0 \tag{23}$$

whenever the decomposition equation (14) admits a solution on the unit circle unless both $P_M(j\omega,\tau)$ and $P_0(j\omega,\tau)$ vanish. We will refer to (23) as the *Orlando magnitude equation.*

Suppose that either (19) or (21) is satisfied. It is still possible for the Orlando magnitude equation (23) to be satisfied even if the decomposition equation (14) has no solution on the unit circle. This case arises when there are two solutions x_i and x_k of (14) (when $P_M(\omega,\tau) \neq 0$) or (15) (when $P_0(\omega,\tau) \neq 0$) that are on the same radial line and their magnitudes are reciprocal of each other:

$$|x_i| = 1/|x_k|, \tag{24}$$

$$\angle x_i = \angle x_k + r2\pi, \quad r \text{ integer}. \tag{25}$$

The above can be easily seen from (20) and (22).

We will introduce some assumptions in order to avoid some degenerate cases and present our approach within a limited amount of space. The cases that violate one or more assumptions can usually be treated individually based on the same principle presented here. An example will be presented in Section 5.

Assumption I. For all $\tau \in \mathcal{T}$,

$$\deg(P_{0\tau}) = n, \tag{26}$$

$$\deg(P_{k\tau}) \leq n, \ k = 1, 2, \cdots, M. \tag{27}$$

Furthermore, all the roots of the following equation

$$s^M + c_{1\tau}s^{M-1} + c_{2\tau}s^{M-2} + \cdots + c_{M-1,\tau}s + c_M = 0 \tag{28}$$

satisfy $|s| < 1$, where

$$c_{k\tau} = \lim_{\lambda\to\infty} \frac{P_{k\tau}(\lambda)}{P_{0\tau}(\lambda)}, \ k = 1, 2, \ldots, M.$$

Assumption II. No $(\omega, \tau) \in \mathbb{R} \times \mathcal{T}$ simultaneously satisfies

$$P_M(j\omega, \tau) = 0, \tag{29}$$

and

$$P_0(j\omega, \tau) = 0. \tag{30}$$

Assumption III. Any $(\omega^*, \tau^*) \in$ in $\mathbb{R}_+ \times \mathcal{T}$ that satisfies (13) must also satisfy

$$\partial_\lambda D(\lambda, \tau)|_{\lambda=j\omega^*, \tau=\tau^*} \neq 0. \tag{31}$$

Furthermore, let $\lambda(\tau)$ be the unique function implicitly defined by (1) in a sufficiently small neighborhood of τ^* taking value within a small neighborhood of $j\omega^*$, then

$$\mathcal{R}(\lambda(\tau)) \neq 0 \quad \text{for all } \tau \neq \tau^* \tag{32}$$

in this small neighborhood.

Assumption IV. There are only a finite number of $(\omega, \tau) \in \mathbb{R}_+ \times \mathcal{T}$ that simultaneously satisfy the Orlando magnitude equation (23) and the following equation

$$\partial_\omega F(\omega, \tau) = 0. \tag{33}$$

In the remaining part of this paper, these four nondegeneracy assumptions will be assumed to be satisfied unless specifically pointed out.

The above four assumptions parallel those for the single delay case given in [20]. Indeed, Assumption I assures that τ-decomposition is applicable. In this assumption, (28) is the difference equation associated with the system (1). When the system is of retarded type [12, 17], (27) is satisfied with strict inequality because $c_{k\tau} = 0$, $k = 1, 2, \ldots M$, and the assumption is automatically satisfied. When the system is of neutral type [12, 17], the assumption implies that there exists an $\varepsilon > 0$ such that only a finite number of characteristic roots of the system (1) are on the right hand side of the vertical line $\mathcal{R}(\lambda) = -\varepsilon$, and they are continuous functions of τ [12]. It should be pointed out that the assumption $P_k(\lambda, \tau)$, $k = 0, 1, \ldots, M$ be continuous with respect to τ is essential. Indeed, [28] gives an example that a characteristic root may suddenly appear on the positive real axis when the characteristic quasipolynomial depends on $\frac{1}{\tau}$ (and thus discontinuous at $\tau = 0$).

Assumption II implies that the Orlando magnitude equation (23) is satisfied for all the pairs (ω, τ) such that the decomposition equation (14) has at least one root on the unit circle. Violation of this assumption would require two real variables ω and τ to satisfy two complex equations (29) and (30), which are equivalent to four real equations. Therefore, it is indeed rare that this assumption would be violated.

Assumption III means that all imaginary roots of $D_\tau(\lambda)$ are simple, and the curve $\lambda(\tau)$ intersects with the imaginary axis only at one point $\lambda(\tau^*)$ in a small neighborhood of τ^*. When $D(\lambda, \tau)$ is differentiable with respect to τ, then a sufficient condition for

the assumption is

$$\mathcal{R}(\lambda'(\tau^*)) \neq 0. \tag{34}$$

Assumption IV is a very minor restriction. Similar to the single delay case, $F(\omega, \tau)$ is an even polynomial function of ω for any given τ. If the pair (ω, τ) satisfies both (23) and (33), then ω is a multiple root of this polynomial.

The following theorem plays an important role in simplifying stability analysis.

Theorem 1. *Let $(\omega, \tau) \in \mathbb{R} \times \mathcal{T}$ satisfy the Orlando magnitude equation (23) and the following equation*

$$\partial_\omega F(\omega, \tau) \neq 0. \tag{35}$$

Then, there is one and only one root of $\Phi_{\omega\tau}(x)$ on the unit circle, and this root is simple.

Proof. Let $(\omega^*, \tau) \in \mathbb{R} \times \mathcal{T}$ satisfy (23) and assume $P_M(j\omega^*, \tau) \neq 0$. It is sufficient to show that (35) must be violated if $\Phi_{\omega^*\tau}(x)$ has either one of the following two types of roots:

I) At least two roots are on the unit circle;

II) Two roots x_i, x_k that are not on the unit circle, and satisfy (24) and (25).

We will first show Case I). This case can be further divided to two subcases:

Ia) At least two simple roots are on the unit circle;

Ib) There is a multiple root on the unit circle.

For Case Ia), according to the equation (20) in Lemma 2, we may write

$$F(\omega, \tau) = R(\omega, \tau)(1 - x_i\bar{x}_i)(1 - x_k\bar{x}_k), \tag{36}$$

where $R(\omega, \tau)$ is continuous, and x_i and x_k are continuously differentiable with respect to ω at ω^*. Furthermore,

$$x_i^* = x_i(\omega^*), x_k^* = x_k(\omega^*)$$

satisfy

$$x_i^*\bar{x}_i^* = x_k^*\bar{x}_k^* = 1. \tag{37}$$

Apply Lemma 4 in Appendix B, we conclude that F is differentiable with respect to ω at ω^*, and

$$\begin{aligned}
\partial_\omega F(\omega^*, \tau) &= R(\omega^*, \tau)\frac{d}{d\omega}[(1 - x_i\bar{x}_i)(1 - x_k\bar{x}_k)]_{x_i=x_i^*, x_k=x_k^*} \\
&= R(\omega^*, \tau)\{(1 - x_i^*\bar{x}_i^*)[-x_k'(\omega^*)\bar{x}_k(\omega^*) - x_k(\omega^*)\bar{x}_k'(\omega^*)] \\
&\quad +[-x_i'(\omega^*)\bar{x}_i(\omega^*) - x_i(\omega^*)\bar{x}_i'(\omega^*)](1 - x_k^*\bar{x}_k^*)\} \\
&= 0
\end{aligned}$$

in view of (37).

For Case Ib), let the root $x = x_0 \in \mathbb{S}$ have multiplicity p when $\omega = \omega^*$. Then, for $\omega = \omega^* + \varepsilon$, the multiple root splits into p roots, and, from the equation (20) in Lemma 2,

$$F(\omega, \tau) = S(\omega, \tau)\prod_{i=1}^{p}\prod_{k=1}^{p}(1 - x_i\bar{x}_k), \tag{38}$$

where $S(\omega,\tau)$ and $x_i, i = 1,2,\dots p$ are continuous. In general, these x_i's are not differentiable with respect to ω at ω^*. Indeed[25, 26], they can be expressed using puiseux series as

$$x_k = x_0 + c_k\varepsilon^{1/p} + o(\varepsilon^{1/p}), \quad k = 1,2,\dots p, \tag{39}$$

where $c_k, k = 1,2,\dots,p$ are complex constants. It can be calculated that,

$$\begin{aligned} 1 - x_i\bar{x}_k &= 1 - (x_0 + c_i\varepsilon^{1/p} + o(\varepsilon^{1/p}))\overline{(x_0 + c_k\varepsilon^{1/p} + o(\varepsilon^{1/p}))} \\ &= 1 - x_0\bar{x}_0 - (x_0\bar{c}_k + \bar{x}_0c_i)\varepsilon^{1/p} + o(\varepsilon^{1/p}) \\ &= -(x_0\bar{c}_k + \bar{x}_0c_i)\varepsilon^{1/p} + o(\varepsilon^{1/p}). \end{aligned}$$

Therefore,

$$\begin{aligned} F(\omega,\tau) &= S(\omega,\tau)\prod_{i=1}^{p}\prod_{k=1}^{p}(1 - x_i\bar{x}_k) \\ &= S(\omega,\tau)(c_\Pi\varepsilon^p + o(\varepsilon^p)), \end{aligned}$$

where

$$c_\Pi = \prod_{i=1}^{p}\prod_{k=1}^{p}(-x_0\bar{c}_k - \bar{x}_0c_i). \tag{40}$$

As a result,

$$\partial_\omega F(\omega,\tau) = \lim_{\varepsilon\to 0}\frac{S(\omega,\tau)(c_\Pi\varepsilon^p + o(\varepsilon^p))}{\varepsilon} = 0.$$

For case II), we will only show the case where both x_i and x_k are simple. The case with multiple roots can be shown in a way similar to Case Ib). Similar to the case Ia), we may again write, from the equation (20) in Lemma 2,

$$F(\omega,\tau) = Q(\omega,\tau)(1 - x_i\bar{x}_k)(1 - x_k\bar{x}_i), \tag{41}$$

where $Q(\omega,\tau)$ is continuous, and x_i and x_k are differentiable at $\omega = \omega^*$, and $x_i(\omega^*) = x_i^*, x_k(\omega^*) = x_k^*$ satisfy

$$x_i^*\bar{x}_k^* = x_k^*\bar{x}_i^* = 1. \tag{42}$$

Apply Lemma 4 in Appendix B, we again conclude that F is differentiable with respect to ω at ω^*, and

$$\begin{aligned} \partial_\omega F(\omega^*,\tau) &= Q(\omega^*,\tau)\{(1 - x_i^*\bar{x}_k^*)[-x_k'(\omega^*)\bar{x}_i(\omega^*) - x_k(\omega^*)\bar{x}_i'(\omega^*)] \\ &\quad + [-x_i'(\omega^*)\bar{x}_k(\omega^*) - x_i(\omega^*)\bar{x}_k'(\omega^*)](1 - x_k^*\bar{x}_i^*)\} \\ &= 0. \end{aligned}$$

Finally, if $P_M(j\omega,\tau) = 0$, then $P_0(j\omega,\tau) \neq 0$ per Assumption II. The proof proceeds similarly using the expression (22) instead of (20). □

In the following examples, the equation (23) is satisfied, but the condition (35) in Theorem 1 is violated for the given ω and τ. According to the theorem above, $\Phi_{\omega\tau}(x)$ cannot have one and only one simple root on the unit circle. They illustrate a number of possible scenarios.

Example 1. *Consider the characteristic equation (1) with*

$$P_0 = -0.6\tau + 0.8\lambda,\ P_1 = -0.9\tau - 1.8\lambda,\ P_2 = 1.$$

At $\omega = 1$ and $\tau = 1$, $\Phi_{\omega\tau}(x)$ has no root on the unit circle. However, it has two roots $x_1 = 0.4 + 0.8j$ and $x_2 = 0.5 + j$ satisfying (24) and (25).

Example 2. *Consider the characteristic equation (1) with*

$$P_0 = -2\tau,\ P_1 = 1 - \lambda\tau,\ P_2 = \lambda\tau.$$

At $\omega = 2$ and $\tau = 0.5$, $\Phi_{\omega\tau}(x)$ has two distinct simple roots $x_1 = 1$ and $x_2 = j$ on the unit circle.

Example 3. *Consider the characteristic equation (1) with*

$$P_0 = \lambda\tau,\ P_1 = -\sqrt{2} - \sqrt{2}\lambda\tau,\ P_2 = 0.64\tau^2.$$

At $\omega = 0.8$ and $\tau = 1.25$, $\Phi_{\omega\tau}(x)$ has a double root $x_1 = x_2 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$ on the unit circle.

3 Stability analysis

Due to Theorem 1, the general process of stability analysis for systems with commensurate delay can be made parallel to the one for the single delay case. The first step of the process is to determine the critical delay corresponding to the imaginary characteristic roots. These critical delays divide the delay interval of interest $\mathcal{T}$ into subintervals. The number of characteristic roots on the RHP is fixed for τ to be within each such subinterval. The second step is to determine the increment of RHP roots at each critical delay. This will allow us to determine the number of RHP roots for each delay subinterval, especially, those delay subintervals without RHP roots can be identified, and the stability analysis is complete.

3.1 Determination of critical delays

The first step of our analysis is to identify all τ values such that the characteristic equation has at least one imaginary root.

Recall that if $\lambda = j\omega$ is an imaginary root of $D_\tau(\lambda)$, ω must be a real root of $F_\tau(\omega)$. Therefore it is natural to expect that the real roots of $F_\tau(\omega)$ will play an important role in the analysis. We first show that $\mathcal{T}$ can be decomposed into disjoint intervals in which the number of real roots of $F_\tau(\omega)$ is invariant.

Let $\tau^{(i)} \in \mathcal{T}, i = 1, 2, \ldots K-1$ be the set of $\tau \in \mathcal{T}$ such that the equations (23) and (33) are simultaneously satisfied by such an τ and some $\omega \geq 0$ (as $F_\tau(\omega)$ is an

even function of ω, it is sufficient to restrict ourselves to studying $\omega \geq 0$). Arrange this set in ascending order,

$$\tau^{(1)} < \tau^{(2)} < \cdots < \tau^{(K-1)},$$

and write $\tau^{(0)} = \tau^l$, $\tau^{(K)} = \tau^u$. Then, the interval $\mathcal{T}$ can be divided into K subintervals

$$\mathcal{T}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}], \ i = 1, \ldots, K. \tag{43}$$

Each subinterval $\mathcal{T}^{(i)}$ is known as a continuous frequency curve interval (CFCI). It has been shown in Proposition 1 of [20] that, for all τ within $\mathcal{T}_o^{(i)}$, the interior of the CFCI $\mathcal{T}^{(i)}$, $F_\tau(\omega)$ admits a fixed number of simple real roots. These roots are continuous functions of τ, denoted as $\omega_k^{(i)}(\tau)$, $k = 1, 2, ..., m^{(i)}$, and are known as frequency curves. The definition of these continuous functions (and the frequency curves) is extended to $\mathcal{T}^{(i)}$ by requiring them to be continuous at $\tau^{(i-1)}$ and τ^i.

For any $\tau \in \mathcal{T}_o^{(i)}$, Theorem 1 implies that the decomposition equation

$$\Phi_{\omega_k^{(i)}(\tau),\tau}(x) = 0 \tag{44}$$

has one and only one solution on the unit circle. This has two important implications: First, the pair $(\omega_k^{(i)}(\tau), \tau)$ satisfies the magnitude condition. Second, there is no need to consider more than one solutions of the decomposition equation (44) on the unit circle.

Let the unique unit circle solution of the decomposition equation (44) be denoted as $z_k^{(i)}(\tau)$, then $z_k^{(i)}(\tau)$ is a continuous function of τ, and we can extend its definition to the entire CFCI by requiring the function $z_k^{(i)}(\tau)$ to be continuous at the two ends of $\mathcal{T}^{(i)}$.

Define

$$\theta_k^{(i)}(\tau) = \angle z_k^{(i)}(\tau) + \omega_k^{(i)}(\tau)\tau, \tag{45}$$

where $\angle z_k^{(i)}(\tau)$ is the phase angel of the complex number $z_k^{(i)}(\tau)$ and is required to be a continuous function of τ in $\mathcal{T}^{(i)}$. Notice, the value of $\angle z_k^{(i)}(\tau)$ is not necessarily restricted to any 2π interval due to continuity requirement.

Using Lemma 1, we obtain the following result immediately.

Corollary 1. *For any given i, a pair $(\omega^*, \tau^*) \in \bar{\mathbb{R}}_+ \times \mathcal{T}^{(i)}$ satisfies (13) if and only if there exist some index k such that $\omega^* = \omega_k^{(i)}(\tau^*)$ and*

$$\theta_k^{(i)}(\tau^*) = 2r\pi \tag{46}$$

for some integer r.

Going through each interval $\mathcal{T}^{(i)}$ and traversing through each curve $\omega_k^{(i)}(\tau)$, $k = 1, 2, \ldots, m^{(i)}$, within $\mathcal{T}^{(i)}$, we may identify all $\tau^* = \tau_l$, $l = 1, 2, \ldots, L$ such that (46) holds for some integer r and indices k and i. Each such τ_l is known as a critical

delay. For each given critical delay τ_l, it is possible that (46) is satisfied for more than one k. Denote these k's as $k_h, h = 1, 2, \ldots, H_l$. Then we denote the corresponding $\omega_{k_h}^{(i)}(\tau_l) \geq 0$ as ω_{lh}, $h = 1, 2, \ldots, H_l$. In this way we can identify all the pairs (ω_{lh}, τ_l), $h = 1, 2, \ldots, H_l; l = 1, 2, \ldots, L$, that satisfy (13).

3.2 Determination of stable delay intervals

We now describe the second step of stability analysis mentioned at the beginning of the section. Under Assumption III, Equation (1) defines an implicit function $\lambda(\tau)$ in a small neighborhood of τ_l, and $\lambda(\tau)$ lies in a small neighborhood of $j\omega_{lh}$, and satisfies (32) for $\tau \neq \tau_l$.

If $\tau_l \neq \tau^l$, define

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \lim_{\epsilon \to 0^+} \frac{\mathrm{sgn}\left(\mathcal{R}\left(\lambda(\tau_l + \epsilon)\right)\right) - \mathrm{sgn}\left(\mathcal{R}\left(\lambda(\tau_l - \epsilon)\right)\right)}{2}. \tag{47}$$

Due to Assumption III, possible values of $\mathrm{Inc}(\omega_{lh}, \tau_l)$ are $-1, 0$, and 1. When $\mathrm{Inc}(\omega_{lh}, \tau_l) = 1$, a characteristic root moves from left half-plane (LHP) to RHP through $j\omega_{lh}$ on the imaginary axis. The movement is in the opposite direction if $\mathrm{Inc}(\omega_{lh}, \tau_l) = -1$. The root touches the imaginary axis and returns to the same side of the imaginary axis without crossing it if $\mathrm{Inc}(\omega_{lh}, \tau_l) = 0$.

If $\tau_l = \tau^l$, which implies $\tau_l = \tau_1$, define instead

$$\mathrm{Inc}(\omega_{1h}, \tau_1) = \max\{0, \lim_{\epsilon \to 0^+} \mathrm{sgn}\left(\mathcal{R}\left(\lambda(\tau_1 + \epsilon)\right)\right)\}. \tag{48}$$

Now let the number of RHP roots of $D_\tau(\lambda)$ be $N^u(\tau)$. For any $\tau \in \mathcal{T}$, $\tau \neq \tau_l$, $l = 1, 2, \ldots, L$, it is easy to see that the following relation holds:

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L_\tau} \mathrm{Inc}(\tau_l), \tag{49}$$

where $L_\tau = \max\{l \mid \tau_l < \tau\}$ and

$$\mathrm{Inc}(\tau_l) = 2 \sum_{h=1}^{H_l} \mathrm{Inc}(\omega_{lh}, \tau_l). \tag{50}$$

The coefficient 2 in the above equation is due to the fact that the crossing direction is the same at $j\omega_{lh}$ and $-j\omega_{lh}$. The subintervals of $\mathcal{T}$ corresponding to $N^u(\tau) = 0$ can be determined. These subintervals of $\mathcal{T}$ give the set of τ corresponding to stable system.

4 Crossing direction analysis

In the last section, we have described a process of determining the set of τ for the system to be asymptotically stable within the given interval of interest $\mathcal{T} = [\tau^l, \tau^u]$

provided that $\text{Inc}(\omega^*, \tau^*)$ can be determined for any given critical pair (ω^*, τ^*). A critical pair is one that satisfies

$$D(j\omega^*, \tau^*) = 0. \tag{51}$$

In general, $\text{Inc}(\omega^*, \tau^*)$ may be determined by a numerical method. It is, however, obviously of interest to use an analytical method. This is the focus of this section. For this purpose, we first replace Assumption III by one that can be easily verified analytically.

Assumption IIIa. The pair (ω^*, τ^*) satisfies (51), $D(\lambda, \tau)$ is differentiable with respect to τ up to Nth order in a neighborhood of $(j\omega^*, \tau^*)$. Furthermore,

$$\partial_\omega F(\omega^*, \tau^*) \neq 0. \tag{52}$$

It should be pointed out that Assumption IIIa is not equivalent to Assumption III. For the single delay case, it was shown in Lemma 1 of [20] that the last part of Assumption IIIa implies the first part of Assumption III. The following lemma shows that this is also true for the commensurate delay case.

Lemma 3. *Suppose (ω^*, τ^*) satisfies (51), and $D(\lambda, \tau)$ is differentiable with respect to τ in a neighborhood of $(j\omega^*, \tau^*)$. If it satisfies the condition (52) in Assumption IIIa, then it also satisfies the condition (31) in Assumption III.*

Proof. It is sufficient to show that if (31) is not satisfied, then (52) is not satisfied. In other words, we only need to show that

$$\partial_\lambda D(j\omega^*, \tau^*) = 0, \tag{53}$$

implies

$$\partial_\omega F(\omega^*, \tau^*) = 0. \tag{54}$$

Let (53) be satisfied. First, assume $P_M(j\omega^*, \tau^*) \neq 0$. Consider the decomposition equation

$$\Phi_{\omega\tau^*}(x) = \sum_{k=0}^{M} P_k(j\omega, \tau^*)x^k = 0. \tag{55}$$

The equation (51) means that, for $\omega = \omega^*$, the polynomial equation (55) has a solution $x^* = e^{-j\omega^*\tau^*}$. If x^* is a multiple root of $\Phi_{\omega^*\tau^*}(x)$, then (54) is satisfied according to Theorem 1. It remains to be shown that (54) is satisfied when x^* is a simple root of $\Phi_{\omega^*\tau^*}(x)$.

For ω sufficiently close to ω^*, let

$$\varepsilon = -D(j\omega, \tau^*) = -\Phi_{\omega\tau^*}(e^{-j\omega\tau^*}). \tag{56}$$

Then, (51) and (53) imply that $j\omega^*$ is a multiple root of $D_{\tau^*}(j\omega)$. It follows that

$$\lim_{\omega\to\omega^*} \frac{\varepsilon}{\omega - \omega^*} = \left.\frac{d\varepsilon}{d\omega}\right|_{\omega=\omega^*} = -j\left.\frac{dD_{\tau^*}}{d\lambda}\right|_{\lambda=j\omega^*} = 0. \tag{57}$$

Equation (56) can be written as

$$P_0(j\omega,\tau^*)+\varepsilon+\sum_{k=1}^{M}P_k(j\omega,\tau^*)\hat{x}^k=0, \tag{58}$$

where

$$\hat{x}=e^{-j\omega\tau^*}. \tag{59}$$

Comparing the equations (58) and (55), the two polynomial equations differ by a small parameter ε in the constant term. Therefore, there is a solution of (55)

$$x=\hat{x}+\delta, \tag{60}$$

where δ is small. Furthermore, as x is a simple root of $\Phi_{\omega\tau^*}(x)$, δ is also differentiable with respect to ε. Therefore

$$\lim_{\varepsilon\to 0}\left|\frac{\delta}{\varepsilon}\right|<\infty. \tag{61}$$

Applying Lemma 2 to $\Phi_{\omega^*\tau^*}(x^*)$ and $\Phi_{\omega\tau^*}(x)$, we obtain

$$F(\omega^*,\tau^*)=0, \tag{62}$$

and

$$F(\omega,\tau^*)=Q(\omega,\tau^*)(1-x\bar{x}), \tag{63}$$

where $Q(\omega,\tau^*)$ is continuous. But

$$\begin{aligned}1-x\bar{x}&=1-(\hat{x}+\delta)\overline{(\hat{x}+\delta)}\\&=1-|\hat{x}|^2-\hat{x}\bar{\delta}-\overline{(\hat{x}\bar{\delta})}-\delta\bar{\delta}\\&=-\hat{x}\bar{\delta}-\overline{(\hat{x}\bar{\delta})}-\delta\bar{\delta}.\end{aligned} \tag{64}$$

Therefore,

$$\begin{aligned}\partial_\omega F(\omega^*,\tau^*)&=\lim_{\omega\to\omega^*}\frac{-Q(\omega,\tau)\left[\hat{x}\bar{\delta}+\overline{(\hat{x}\bar{\delta})}+\delta\bar{\delta}\right]}{\varepsilon}\frac{\varepsilon}{\omega-\omega^*}\\&=0\end{aligned} \tag{65}$$

in view of (61) and (57). In other words, (54) is satisfied.

If $P_M(j\omega^*,\tau^*)=0$, then $P_0(j\omega^*,\tau^*)\neq 0$ due to Assumption II. We may consider $\Psi_{\omega\tau^*}(x)$ instead of $\Phi_{\omega\tau^*}(x)$, and the proof proceeds similarly. □

Under Assumption IIIa, the above lemma indicates that the equation

$$D(\lambda,\tau)=0 \tag{66}$$

defines a unique function $\lambda(\tau)$ when τ is in a sufficiently small neighborhood of τ^*, and λ is restricted to a small neighborhood of $j\omega^*$. This function is differentiable up to Nth order. Furthermore, if for some $m \leq N$,

$$\mathcal{R}\left(\frac{d^k\lambda(\tau^*)}{d\tau^k}\right) = 0, \quad k = 1, 2, \dots m-1, \tag{67}$$

$$\mathcal{R}\left(\frac{d^m\lambda(\tau^*)}{d\tau^m}\right) \neq 0, \tag{68}$$

and $\tau^* > \tau^l$, then

$$\mathrm{Inc}(\omega^*, \tau^*) = \begin{cases} \mathrm{sgn}\left(\mathcal{R}\left(\frac{d^m\lambda(\tau^*)}{d\tau^m}\right)\right), & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases} \tag{69}$$

If $\tau^* = \tau^l$ instead, then

$$\mathrm{Inc}(\omega^*, \tau^*) = \max\left\{0, \mathrm{sgn}\left(\mathcal{R}\left(\frac{d^m\lambda(\tau^*)}{d\tau^m}\right)\right)\right\}. \tag{70}$$

In view of the above discussions, calculating m and $\mathrm{sgn}\left(\mathcal{R}\left(\frac{d^m\lambda(\tau^*)}{d\tau^m}\right)\right)$ is critical in the determination of the crossing direction of characteristic roots. In [20], the case with single delay and $m = 1$ was solved. The following theorem generalizes the result to the commensurate delay case with $m \geq 1$.

Theorem 2. *Let Assumption IIIa be satisfied. For all τ in a sufficiently small neighborhood of τ^*, let $(\omega(\tau),\theta(\tau))$ be the unique solution in a small neighborhood of $(\omega^*, 2r\pi)$ to the equation*

$$\Phi(\omega, \tau, e^{j(\theta-\omega\tau)}) = 0, \tag{71}$$

and let $\lambda(\tau)$ be the unique solution in a small neighborhood of $j\omega^*$ to the equation (66). If

$$\frac{d^k\theta(\tau^*)}{d\tau^k} = 0, \quad k = 1, 2, \dots m-1, \tag{72}$$

$$\frac{d^m\theta(\tau^*)}{d\tau^m} \neq 0, \tag{73}$$

then,

$$\left(\mathcal{R}\left(\frac{d^k\lambda(\tau^*)}{d\tau^k}\right)\right) = 0, \quad k = 1, 2, \dots m-1, \tag{74}$$

$$\mathrm{sgn}\left(\mathcal{R}\left(\frac{d^m\lambda(\tau^*)}{d\tau^m}\right)\right) = (-1)^{N_x(\tau^*)}\,\mathrm{sgn}\left(\frac{\partial F(\omega^*, \tau^*)}{\partial\omega}\right)\mathrm{sgn}\left(\frac{d^m\theta(\tau^*)}{d\tau^m}\right), \tag{75}$$

where $N_x(\tau)$ is the number of roots of the polynomial $\Phi_{\omega(\tau)\tau}(x)$ outside the unit circle if $P_M(j\omega(\tau), \tau) \neq 0$, or the number of roots of the polynomial $\Psi_{\omega(\tau)\tau}(x)$ inside the unit circle if $P_0(j\omega(\tau), \tau) \neq 0$.

The proof of this theorem will be given in Appendix D. The theorem indicates that if the devatives of $\theta(\tau)$ vanish up to the $(m-1)$th order, then the real part of the derivatives of the critical characteristic roots also vanish up to the $(m-1)$th order. Furthermore, the sign of the lowest-order nonzero derivative of $\theta(\tau)$ can be related to sign of the real part of the lowest-order derivative of critical characteristic root with nonzero real part. This means that the crossing direction of a critical characteristic root can be obtained by the crossing direction of the phase curve $\theta(\tau)$, which is formally stated in the following corollary.

Corollary 2. *Let $\tau^* \in \mathcal{T}_o^{(i)} = (\tau^{(i-1)}, \tau^{(i)})$, and $m \leq N$. If $\theta(\tau)$ moves from below the $2r\pi$ line to above it as τ increases through τ^*, then, $\mathrm{Inc}(\omega^*, \tau^*) = (-1)^{N_x(\tau^*)} \operatorname{sgn}\left(\frac{\partial F(\omega^*,\tau^*)}{\partial \omega}\right)$. If $\theta(\tau)$ crosses the $2r\pi$ line from above to below instead, then $\mathrm{Inc}(\omega^*, \tau^*) = (-1)^{N_x(\tau^*)+1} \operatorname{sgn}\left(\frac{\partial F(\omega^*,\tau^*)}{\partial \omega}\right)$.*

Proof. This is obvious from (75) in Theorem 2 and (69). □

The above corollary means that if the curve $\theta(\tau)$ generated in Section III is sufficiently smooth, then the crossing direction (and therefore $\mathrm{Inc}(\omega^*, \tau^*)$) can be obtained by observing the direction that $\theta(\tau)$ crosses the $2r\pi$ line, provided that $N_x(\tau^*)$ and the sign of $\frac{\partial F(\omega^*,\tau^*)}{\partial \omega}$ are available. The following proposition means that these two values need to be checked only at one point in each continuous frequency curve in each CFCI.

Proposition 1. *Within each $\mathcal{T}_o^{(i)}$ and for each continuous frequency curve $\omega(\tau)$ that satisfies $F(\omega(\tau), \tau) = 0$ defined in $\mathcal{T}_o^{(i)}$, $\operatorname{sgn}\left(\frac{\partial F(\omega,\tau)}{\partial \omega}\big|_{\omega=\omega(\tau)}\right)$ and $N_x(\tau)$ remain constant.*

Proof. By the definition of $\mathcal{T}_o^{(i)}$,

$$\frac{\partial F(\omega(\tau), \tau)}{\partial \omega} \neq 0, \quad \tau \in \mathcal{T}_o^{(i)}. \tag{76}$$

To show sign invariance of $\frac{\partial F(\omega,\tau)}{\partial \omega}\big|_{\omega=\omega(\tau)}$, observe that $\frac{\partial F(\omega,\tau)}{\partial \omega}\big|_{\omega=\omega(\tau)}$ is a continuous function of τ, and its sign can change only if it vanishes first, which contradicts (76).

To show the invariance of $N_x(\tau)$, observe that $\Phi_{\omega^*\tau^*}(x)$ has $N_x(\tau^*)$ roots outside of the unit circle, one root on the unit circle, and $M - N_x(\tau^*) - 1$ roots inside the unit circle if $P_M(j\omega(\tau), \tau) \neq 0$. In order for $N_x(\tau)$ to change within $\mathcal{T}_o^{(i)}$, due to continuity of the roots of $\Phi_{\omega(\tau)\tau}(x)$ with respect to τ when $P_M(j\omega(\tau), \tau) \neq 0$, either one of roots outside the unit circle, or one of the roots inside the unit circle must first reach the unit circle for some $\tau = \hat{\tau} \in \mathcal{T}_o^{(i)}$. But this would mean that $\Phi_{\omega(\hat{\tau})\hat{\tau}}(x)$ has two roots at the unit circle, which would require $\frac{\partial F(\omega,\hat{\tau})}{\partial \omega}\big|_{\omega=\omega(\hat{\tau})} = 0$ in view of Theorem 1, thus contradicting (76).

At a τ such that $P_M(j\omega(\tau), \tau) = 0$, then $P_0(j\omega(\tau), \tau) \neq 0$ per Assumption II, the invariance of $N_x(\tau)$ can be shown by considering $\Psi_{\omega(\tau)\tau}(x)$ instead of $\Phi_{\omega(\tau)\tau}(x)$. □

It is also possible to analytically obtain $\frac{d^k\theta(\tau)}{d\tau^k}$, $k = 1, 2, \ldots m$. This can be done by taking the derivatives of the equation $\Phi(\omega, \tau, e^{j(\theta-\omega\tau)}) = 0$,

$$\frac{d^k}{d\tau^k}\Phi(\omega, \tau, e^{j(\theta-\omega\tau)})|_{\tau=\tau^*} = 0, \tag{77}$$

and solving for $\frac{d^k\theta(\tau^*)}{d\tau^k}$ and $\frac{d^k\omega(\tau^*)}{d\tau^k}$. The nonzero $\frac{d^m\theta(\tau^*)}{d\tau^m}$ may be obtained by a recursive process, although the formulation is progressively more tedious as m increases. In Appendix C, it will be shown that this process is always possible and more specifics of this solution process are provided there.

5 Examples

In this section, we will present two examples to illustrate the stability analysis using the results developed in this paper. The coefficients polynomials depend on the delay either in poynomial or exponential form so that it is easier to see some of the steps analytically. In practice, these may be any other functions as long as the continuity or differentiability conditions are satisfied. We also present an example of stability analysis when some of the nondegeneracy assumptions are violated.

Example 4. *Consider the characteristic equation (9). By definition,*

$$P_0 = \lambda - \alpha,\ P_1 = e^{\alpha\tau},\ P_2 = e^{2\alpha\tau}.$$

Set the delay interval of interest $\mathcal{T} = [0, 0.8]$ *and* $\alpha = \frac{3}{2}$. *The expression of function* F *can be obtained using (18) as*

$$F = -\omega^4 + a_1\omega^2 + a_2,$$

where

$$\begin{aligned} a_1 &= 2a^4 + a^2 - \frac{9}{2}, \\ a_2 &= -a^8 + a^6 + \frac{15}{2}a^4 + \frac{9}{4}a^2 - \frac{81}{16}, \\ a &= e^{\frac{3}{2}\tau}. \end{aligned}$$

By solving (23) and (33) for $(\omega, \tau) \in \mathbb{R}_+ \times \mathcal{T}$, *we find that* $\mathcal{T}$ *can be decomposed into two CFCIs* $\mathcal{T}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$ *and* $\mathcal{T}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, *where* $\tau^{(0)} = 0$, $\tau^{(1)} \approx 0.4045$, $\tau^{(2)} = 0.8$. *Polynomial* $F_\tau(\omega)$ *has one real solution for* $\tau \in \mathcal{T}_0^{(1)}$ *and two real solutions for* $\tau \in \mathcal{T}_0^{(2)}$. *Consequently the frequency curves* $\omega_1^{(1)}(\tau)$, $\omega_1^{(2)}(\tau)$, $\omega_2^{(2)}(\tau)$ *are well defined in the corresponding CFCIs as plotted in Fig. 1a. The associated phase curves are plotted in Fig. 1b. The graph of* $\theta_1^{(1)}$ *crosses the horizontal line* 0 *at* $\tau_1 \approx 0.2368$ *and the graphs of* $\theta_2^{(2)}$ *and* $\theta_1^{(2)}$ *cross the horizontal line* 2π *at* $\tau_2 \approx 0.6878$ *and* $\tau_3 \approx 0.6976$ *respectively, all from below to above. By definition, we have* $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 2.9010$, $\omega_{21} = \omega_2^{(2)}(\tau_2) \approx 5.4195$, $\omega_{31} = \omega_1^{(2)}(\tau_3) \approx 10.0461$.

We verify that the following holds:

$$\partial_\omega F(\omega_1^{(1)}(\tau),\tau) < 0,\ N_x(\omega_1^{(1)}(\tau),\tau) = 1,\ \forall \tau \in \mathcal{T}^{(1)},$$
$$\partial_\omega F(\omega_1^{(2)}(\tau),\tau) > 0,\ N_x(\omega_1^{(2)}(\tau),\tau) = 0,\ \forall \tau \in \mathcal{T}^{(2)},$$
$$\partial_\omega F(\omega_2^{(2)}(\tau),\tau) > 0,\ N_x(\omega_2^{(2)}(\tau),\tau) = 0,\ \forall \tau \in \mathcal{T}^{(2)}.$$

Therefore, we conclude from (75) that $Inc(\omega^*,\tau^*) = 1$ *at all the three crossing points. In other words, the imaginary roots* $j\omega_{11}$, $j\omega_{21}$, $j\omega_{31}$ *all move toward the RHP as the delay value increases and sweeps through* τ_1, τ_2, τ_3, *respectively. It can be verified that* $N^u(0) = 0$, *thus the number of unstable roots can be easily computed using (49) and is plotted against* τ *in Fig. 1c. It can thus be concluded that the system (8) is* α*-stable with* $\alpha = 1.5$ *for* $\tau \in [0,\tau_1)$ *and not* α*-stable for* $\tau \in [\tau_1, 0.8]$.

Example 5. *Consider the characteristic equation (1) with the delay interval of interest* $\mathcal{T} = [0, 0.65]$ *and the following* τ*-dependent coefficient polynomials:*

$$P_0 = \lambda^2 + 4,\ P_1 = (1 - 4e^{-2\tau}),\ P_2 = (1 - 2e^{-2\tau})\lambda.$$

The frequency curves $\omega_k^{(i)}(\tau)$ *are plotted in Fig. 2a. The associated phase curves* $\theta_k^{(i)}$ *are plotted in Fig. 2b. The curves* $\theta_2^{(1)}$ *and* $\theta_3^{(2)}$ *cross the horizontal line* 0 *at* $\tau_1 \approx 0.1906$ *and* $\tau_2 \approx 0.5243$, *respectively, both from below to above. The following can be checked:*

$$\partial_\omega F(\omega_2^{(1)}(\tau),\tau) > 0,\ N_x(\omega_2^{(1)}(\tau),\tau) = 1,\ \forall \tau \in \mathcal{T}^{(1)}.$$
$$\partial_\omega F(\omega_3^{(2)}(\tau),\tau) > 0,\ N_x(\omega_3^{(2)}(\tau),\tau) = 0,\ \forall \tau \in \mathcal{T}^{(2)}.$$

Therefore, we can conclude from equation (75) that $Inc(\omega_2^{(1)}(\tau_1),\tau_1) = -1$ *and* $Inc(\omega_3^{(2)}(\tau_2),\tau_2) = 1$. *It can be verified that* $N^u(0) = 2$, *then the number of unstable roots can be easily computed using (49) and is plotted against* τ *in Fig. 2c. We can therefore conclude that this system is stable for* $\tau \in (\tau_1,\tau_2)$.

Example 6. *This example illustrates a degenerate case. Consider the characteristic equation (1) with*

$$P_0 = \lambda^2, P_1 = \lambda + \tau, P_2 = \lambda, P_3 = \lambda,$$

and the delay interval of interest is $\mathcal{T} = [0, 3]$. *The function* $F(\omega,\tau)$ *can be calculated as*

$$F = \omega^{12} - 3\omega^{10} + (2\tau - 4\tau^2)\omega^8 + 9\tau^2\omega^6 - 12\tau^3\omega^4 + 4\tau^4\omega^2.$$

Obviously, the equations (23) and (33) are satisfied for all $\tau \in \mathcal{T}$ *and* $\omega = 0$, *which violates Assumption IV. However, the principles presented in this paper can still be used to analyze this system.*

For $\tau = \tau^l = 0$, $D_\tau(\lambda)$ *has two roots,* -1 *and* 0. *The root at* 0 *moves to the LHP as* τ *increases from* 0 *because*

$$\left.\frac{d\lambda}{d\tau}\right|_{\lambda=0,\tau=0} = -\left.\frac{\frac{\partial D}{\partial \tau}}{\frac{\partial D}{\partial \lambda}}\right|_{\lambda=0,\tau=0} = -1.$$

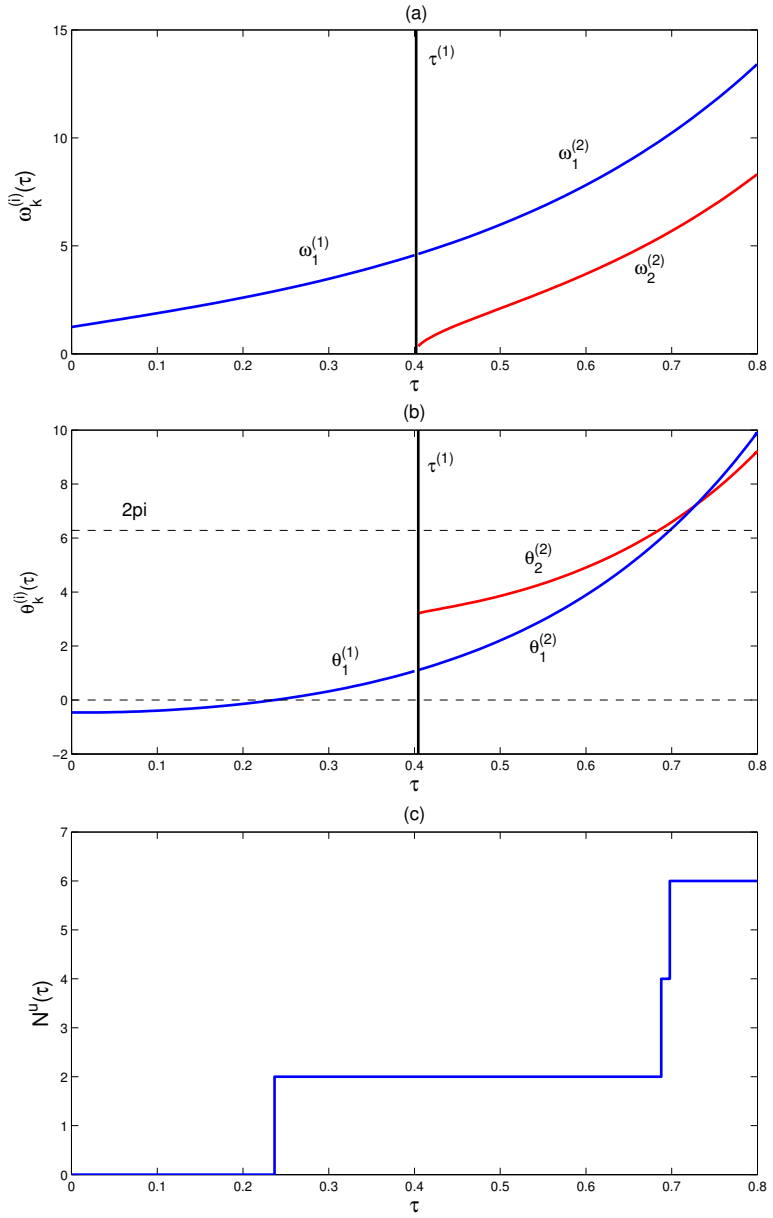

Fig. 1 Plots for Example 4: (a) Frequency curves: There is one frequency curve, $\omega_1^{(1)}(\tau)$, in $\mathcal{T}^1$; and there are two frequency curves, $\omega_1^2(\tau)$ and $\omega_2^2(\tau)$, in $\mathcal{T}^2$. (b) The corresponding phase curves: $\theta_k^{(i)}(\tau)$ corresponds to the frequency curve $\omega_k^{(i)}(\tau)$. (c) The number of roots on the right half-plane, $N^u(\tau)$, for $\tau \in \mathcal{T}$.

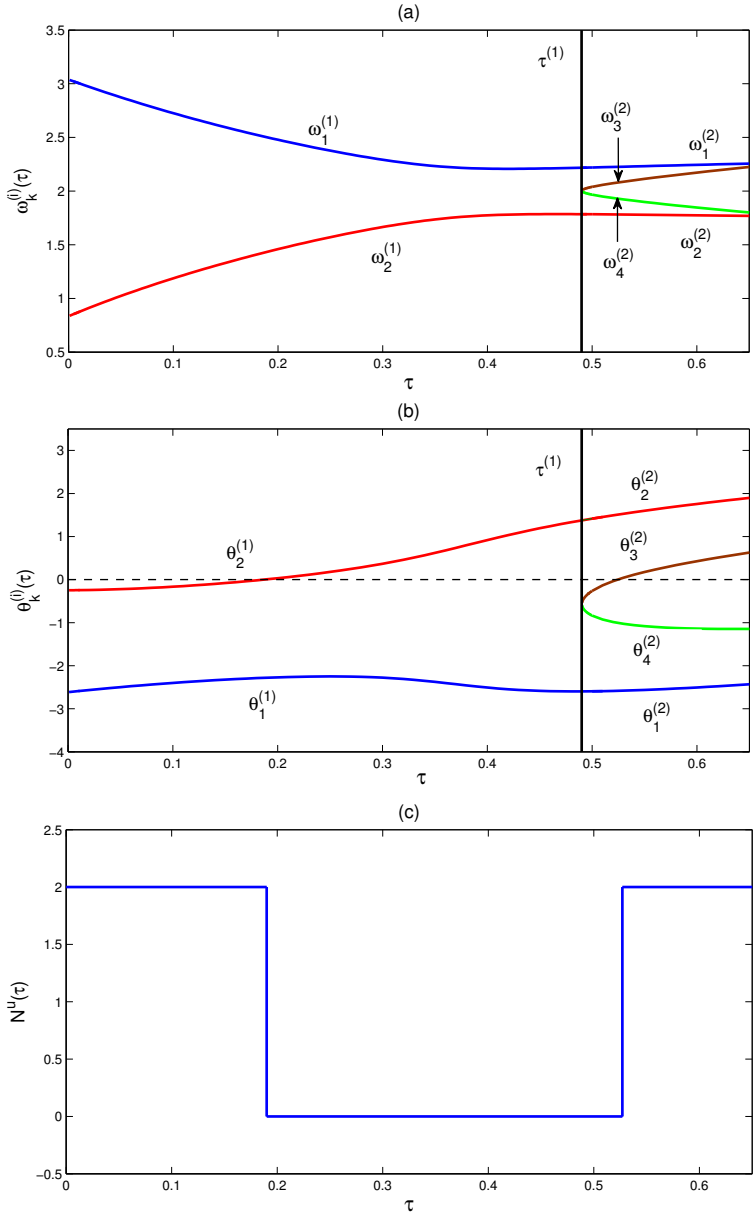

Fig. 2 Plots for Example 5: (a) Frequency curves: There are two frequency curves, $\omega_k^{(1)}(\tau), k = 1, 2$ in $\mathcal{T}^1$; and there are four frequency curves, $\omega_k^{(2)}(\tau), k = 1, 2, 3, 4$, in $\mathcal{T}^2$. (b) The corresponding phase curves: $\theta_k^{(i)}(\tau)$ corresponds to the frequency curve $\omega_k^{(i)}(\tau)$. (c) The number of roots on the right half-plane, $N^u(\tau)$, for $\tau \in \mathcal{T}$.

Therefore, $\mathrm{Inc}(\tau_1) = 0$ *per the equation (70), and the system is stable for* τ *sufficiently small. We may still identify all the crossing points by following the same process as described in Section III. Indeed, for* τ *in the entire interval* $\mathcal{T}$*, there are three real solutions of* $\omega_i(\tau), i = 1, 2, 3$*, and they are plotted in Fig. 3a. For* $\omega_3(\tau) = 0$*, it can be easily calculated that* $\Phi_{\omega\tau}(x)$ *has two roots on the unit circle,* $x = \pm j$*. The corresponding* $\theta_{3\pm} = \pm\pi/2$*. As they do not touch any* $r2\pi$ *line, we can conclude that* $\lambda = 0$ *is not a root for* $\tau > 0$*, which can also be directly verified. For* $\omega_1(\tau)$ *and* $\omega_2(\tau)$*,* $\Phi_{\omega\tau}(x)$ *has a single root on the unit circle, and the corresponding* $\theta_1(\tau)$ *and* $\theta_2(\tau)$ *are plotted in Fig. 3b along with* $\theta_{3\pm}(\tau)$*. It can be seen that* $\theta_1(\tau)$ *crosses* 0 *line at* $\tau_2 \approx 0.5382$*, and then the* 2π *line at* $\tau_3 \approx 2.7206$*, both from below to above. It can be shown that Theorem 2 still applies, and it can be calculated that* $\partial_\omega F(\omega_1(\tau), \tau) > 0$ *and* $N_x(\omega_1(\tau), \tau) = 2$ *at both* $\tau = \tau_2$ *and* τ_3*. Therefore,* $\mathrm{Inc}(\omega_1(\tau_2), \tau_2) = \mathrm{Inc}(\omega_1(\tau_3), \tau_3) = 1$*, and* $N^u(\tau)$ *is plotted in Fig. 3c, from which we conclude that the system is stable for* $\tau = (\tau_1, \tau_2) = (0, 0.5382)$*.*

6 Conclusions and Future Research

A method to analyze the stability of linear time-delay systems represented by differential-difference equations with commensurate delays and delay-dependent coefficients is developed. This method generalizes the previous results for the single delay case. The crossing direction analysis has also been extended to the case that requires higher-order derivatives. The method takes advantage of Orlando's formula. Under some nondegeneracy assumptions parallel to the single delay case, it was shown that the decomposition polynomial has one and only one solution on the unit circle, which allows us to develop a method of systematic stability analysis parallel to the single delay case.

To explore further along this line of research, it is interesting to carry out stability analysis described in this paper for practical systems. Some practical issues such as complexity and amount of calculation may arise. While the assumptions given in this paper are "generic", it would be interesting to see whether these assumptions may be violated in practice, and extensions such as covered in Example 6 may be desirable.

It is also interesing to extend the results discussed here to multiple group of commensurate delays. Such a problem is rather challenging to solve. However, even some special cases would be very interesting. Indeed, when there are two groups with each group consisting of a single delay, a special case was solved in [1].

Appendices

In the appendices, we state the Orlando's formula in Appendix A, which forms an important basis of this paper. Appendix B gives a brief discussion on the derivatives of multivariable functions with dependence between variables. Then we will present a procedure of calculating the derivatives of phase angle with respect to τ, along with some important observations in Appendix C. Finally, we will prove Theorem 2 in Appendix D.

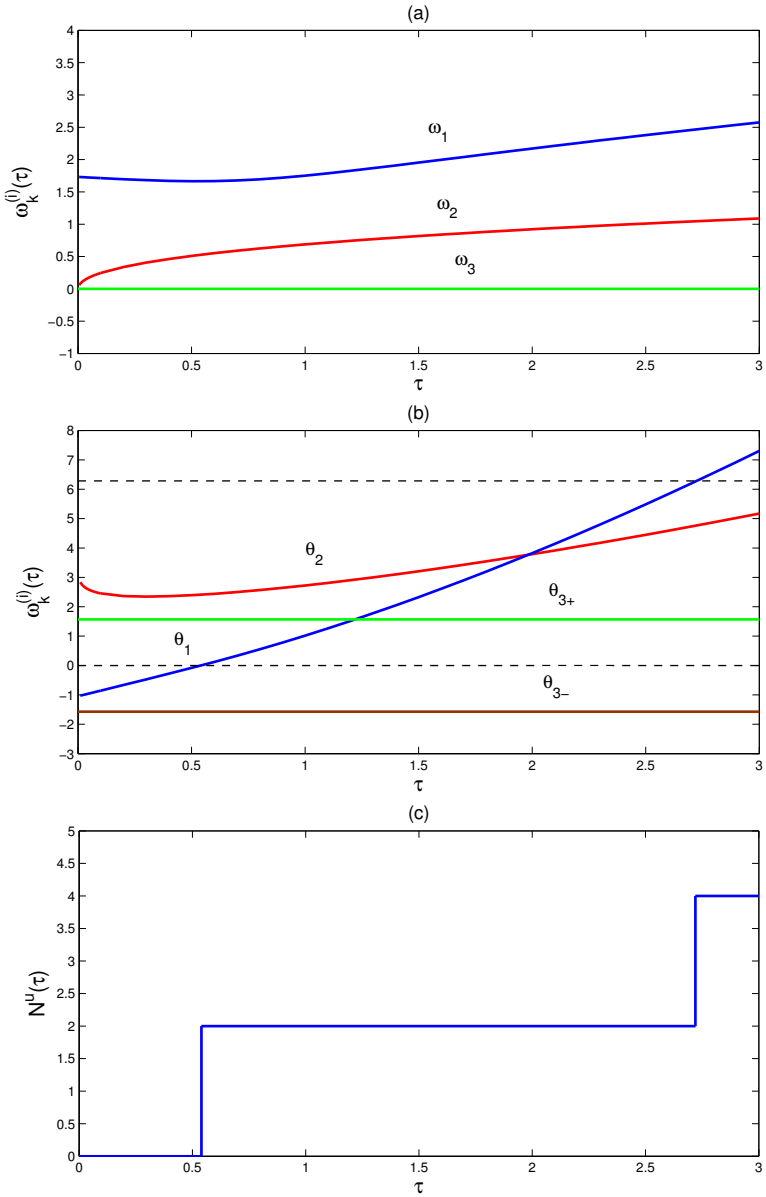

Fig. 3 Plots for Example 6, which is a degenerate case: (a) Frequency curves: There three frequency curves, $\omega_k(\tau), k = 1, 2, 3$ in $\mathcal{T}$, which is not divided in this case. (b) The corresponding phase curves: $\theta_1(\tau)$ and $\theta_2(\tau)$ correspond to $\omega_1(\tau)$ and $\omega_2(\tau)$. Two phase curves, $\omega_{3+}(\tau)$ and $\omega_{3-}(\tau)$ correspond to the frequency curve $\omega_3(\tau)$. (c) The number of roots on the right half-plane, $N^u(\tau)$, for $\tau \in \mathcal{T}$.

Appendix A Orlando's formula

For a polynomial

$$\Gamma(x) = \sum_{k=0}^{M} a_k x^k, \tag{A1}$$

$a_M \neq 0$, let

$$\bar{\Gamma}(x) = \sum_{k=0}^{M} \bar{a}_{M-k} x^k, \tag{A2}$$

and

$$\Xi = \bar{\Gamma}^H(S)\bar{\Gamma}(S) - \Gamma^H(S)\Gamma(S), \tag{A3}$$

where the superscript H represents Hermitian transpose, and S is an $M \times M$ shift matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{A4}$$

Then the following equation is known as the Orlando's formula[22, 33]

$$\det(\Xi) = |a_M|^{2M} \prod_{i=1}^{M} \prod_{j=1}^{M} (1 - x_i \bar{x}_j), \tag{A5}$$

where x_i, $i = 1, 2, \ldots M$ are the roots of the polynomial (A1). An immediate implication of the Orlando's formula is that the polynomial (A1) has a root on the unit circle only if $\det(\Xi) = 0$. This implication is critical in the stability analysis of systems with commensurate delays (with coefficients independent of delays)[6, 7, 18, 19]. Indeed, in stability analysis of such systems, the coefficients a_k become polynomials $P_k(j\omega)$ of frequency. Setting $\det(\Xi)$ to zero allows one to identify the crossing frequencies, one may identify the corresponding critical delays from the roots of $\Gamma(x)$ on the unit circle.

Appendix B Derivatives of multivariable functions

The following lemma is useful in the proof of a number of results in this paper.

Lemma 4. *Let*

$$h(x) = f(x)g(x),$$

where $f(x)$ is continuous and $g(x)$ is continously differentiable in a neighborhood of x^*, and

$$g(x^*) = 0. \tag{B6}$$

Then, $h(x)$ is differentiable at x^*, and

$$h'(x^*) = f(x^*)g'(x^*). \tag{B7}$$

Proof. By definition of derivative and using (B6), we obtain

$$h'(x^*) = \lim_{x\to x^*} \frac{f(x)g(x)}{x-x^*} = \lim_{x\to x^*} f(x)\frac{g(x)-g(x^*)}{x-x^*}$$

from which (B7) is obvious. □

We start with making some observations of derivatives of multivariable functions with variable dependence. The functions are assumed to be continuously differentiable to a sufficiently high order.

Lemma 5. *Consider a function $g(x,\tau)$ of two variables x and τ. If x depends on τ, then*

$$\frac{d^k g}{d\tau^k} = \frac{\partial g}{\partial x}x^{(k)}(\tau) + \frac{\partial^k g}{\partial \tau^k} + U_k, \tag{B8}$$

where $U_k = U_k(x,\tau;x',x'',\ldots,x^{(k-1)})$ is a polynomial of $x'(\tau), x''(\tau),\ldots, x^{(k-1)}(\tau)$ that satisfies

$$U_k(x,\tau;0,0,\ldots,0) = 0. \tag{B9}$$

The coefficients of the polynomial U_k, which are functions of x and τ, are linear combinations of partial derivatives

$$\frac{\partial^{p+q} g}{\partial x^p \partial \tau^q}, \quad 1\le p; 0\le q; 2\le p+q\le k. \tag{B10}$$

The above lemma can be easily shown by induction. The explicit expression of U_k may also be obtained for a small k, for example,

$$U_2 = \frac{\partial^2 g}{\partial x^2}\left[x'(\tau)\right]^2 + 2\frac{\partial^2 g}{\partial x\partial\tau}x'(\tau), \tag{B11}$$

although the expression becomes progressively more tedious as k increases.

Consider now a function of three variables $h(x,y,\tau)$. When y depends on τ, it becomes a function of two variables x and τ. The $(p+q)$th order partial derivative of the function with respect to x and τ considering this dependence is denoted as $\frac{\partial^{p+q}_{y(t)} h}{\partial x^p \partial \tau^q}$. For example,

$$\begin{aligned}
\frac{\partial_{y(\tau)} h}{\partial \tau} &= \frac{\partial h}{\partial y}y'(\tau) + \frac{\partial h}{\partial \tau},\\
\frac{\partial^2_{y(\tau)} h}{\partial x\partial \tau} &= \frac{\partial^2 h}{\partial x\partial y}y'(\tau) + \frac{\partial^2 h}{\partial x\partial \tau},\\
\frac{\partial^2_{y(\tau)} h}{\partial \tau^2} &= \frac{\partial^2 h}{\partial y^2}\left[y'(\tau)\right]^2 + 2\frac{\partial^2 h}{\partial y\partial\tau}y'(\tau) + \frac{\partial h}{\partial y}y''(\tau) + \frac{\partial^2 h}{\partial \tau^2}.
\end{aligned}$$

With the above notation, Lemma 5 may be applied to a function of three variables to obtain the following corollary.

Corollary 3. *Consider a function $h(x, y, \tau)$. If x and y depend on τ, then*

$$\frac{d^k h}{d\tau^k} = \frac{\partial h}{\partial x} x^{(k)}(\tau) + \frac{\partial^k_{y(\tau)} h}{\partial \tau^k} + U_k, \tag{B12}$$

where $U_k = U_k(x, y, \tau; x'(\tau), x''(\tau), \ldots, x^{(k-1)}(\tau))$ is a polynomial of x', x'', $\ldots, x^{(k-1)}$ that satisfies

$$U_k(x, y, \tau; 0, 0, \ldots, 0) = 0. \tag{B13}$$

Furthermore, the term $\frac{\partial^k_{y(\tau)} h}{\partial \tau^k}$ may be expressed as

$$\frac{\partial^k_{y(\tau)} h}{\partial \tau^k} = \frac{\partial h}{\partial y} y^{(k)}(\tau) + \frac{\partial^k h}{\partial \tau^k} + V_k, \tag{B14}$$

where $V_k = V_k(x, y, \tau; y', y'', \ldots, y^{(k-1)})$ is a polynomial of $y', y'', \ldots, y^{(k-1)}$ that satisfies

$$V_k(x, y, \tau; 0, 0, \ldots, 0) = 0. \tag{B15}$$

Appendix C Derivatives of phase angle

With the notation in Appendix B, we will now discuss the calculation of the derivative $\frac{d^k \theta}{d\tau^k}$ in a neighborhood of τ^*. For this purpose, define

$$T(\theta, \omega, \tau) = \Phi(\omega, \tau, e^{j(\theta - \omega\tau)}), \tag{C16}$$

$$W(\omega, \tau) = \Phi(\omega, \tau, e^{-j\omega\tau}). \tag{C17}$$

Then $\theta(\tau)$ and $\omega(\tau)$ satisfy $\theta(\tau^*) = 2r\pi$, $\omega(\tau^*) = \omega^*$, and

$$T(\theta, \omega, \tau) = 0. \tag{C18}$$

Taking kth order derivative of the above equation and applying Corollary 3, yield

$$\begin{aligned} \left(\frac{d^k T}{d\tau^k}\right)_* &= \left(\frac{\partial T}{\partial \theta}\right)_* \left(\frac{d^k \theta}{d\tau^k}\right)_* + \left(\frac{\partial W}{\partial \omega}\right)_* \left(\frac{d^k \omega}{d\tau^k}\right)_* \\ &\quad + (V_{Wk})_* + (U_{Tk})_* \\ &= 0, \end{aligned} \tag{C19}$$

where $(\cdot)_*$ means that the expression is evaluated at $\tau = \tau^*, \omega = \omega^*$ and $\theta = 2r\pi$, V_{Wk} is a polynomial of $\omega'(\tau), \omega''(\tau), \ldots, \omega^{(k-1)}(\tau)$, and U_{Tk} is a polynomial of $\theta'(\tau), \theta''(\tau), \ldots, \theta^{(k-1)}(\tau)$. In arriving at the above expression, we have used the fact that $T(\omega, \tau)|_{\theta = 2r\pi} = W(\omega, \tau)$.

Suppose

$$\left(\frac{d^i \theta}{d\tau^i}\right)_* = 0, \ i = 1, 2, \ldots, k-1. \tag{C20}$$

Then

$$(U_{Tk})_* = 0. \tag{C21}$$

Therefore, we can solve (C19) for $\left(\frac{d^k\theta}{d\tau^k}\right)_*$ by multiplying $\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}$ and taking the imaginary part

$$\left(\frac{d^k\theta}{d\tau^k}\right)_* = -\frac{\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}(V_{Wk})_*\right)}{\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}\left(\frac{\partial T}{\partial \theta}\right)_*\right)}. \tag{C22}$$

Similarly, $\left(\frac{d^k\omega}{d\tau^k}\right)_*$ may be solved from (C19) by multiplying $\overline{\left(\frac{\partial T}{\partial \theta}\right)_*}$ and taking the imaginary part

$$\left(\frac{d^k\omega}{d\tau^k}\right)_* = -\frac{\mathcal{I}\left(\overline{\left(\frac{\partial T}{\partial \theta}\right)_*}(V_{Wk})_*\right)}{\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_*\overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right)}. \tag{C23}$$

The equations (C22) and (C23) may be used starting from $k = 1$ and increasing k until we reach

$$\left(\frac{d^k\theta}{d\tau^k}\right)_* \neq 0,$$

which means that m in Theorem 2 has been reached.

In order for the equations (C22) and (C23) to be valid, the denominator needs to be nonzero. In order to prove this, we need the following lemma.

Lemma 6. *Let Assumption IIIa be satisfied, and let $x_1(\omega)$ be a root of $\Phi_{\omega\tau^*}(x)$ in a small neighborhood of $x_1^* = e^{-j\omega^*\tau^*}$ for ω in a small neighborhood of ω^*. Then $x_1(\omega)$ is differentiable and satisfies*

$$\mathcal{R}(x_1'(\omega^*)\bar{x}_1^*) \neq 0. \tag{C24}$$

Furthermore,

$$\operatorname{sgn}\left(\mathcal{R}(x_1'(\omega^*)\bar{x}_1^*)\right) = (-1)^{N_x(\tau^*)}\operatorname{sgn}\left(\partial_\omega F(\omega^*,\tau^*)\right). \tag{C25}$$

Proof. Assume $P_M(j\omega^*,\tau^*) \neq 0$. Then, we can write

$$F(\omega,\tau^*) = R(\omega)(1 - x_1(\omega)\bar{x}_1(\omega)), \tag{C26}$$

where

$$R(\omega) = -\left|P_M(j\omega,\tau)\right|^{2M}\left(\prod_{i=2}^{M}\chi_{ii}\right)\left(\prod_{k=1}^{M-1}\prod_{i=k+1}^{M}\chi_{ki}\bar{\chi}_{ki}\right), \tag{C27}$$

$$\chi_{ki} = 1 - x_k(\omega)\bar{x}_i(\omega), \quad k,i = 1,2,\ldots,M,$$

and $x_k(\omega)$, $k = 2,3,\ldots,M$ are the remaining $M-1$ roots of $\Phi_{\omega\tau^*}(x)$. Applying Theorem 1, we conclude that $x_1(\omega)$ is differentiable, $R(\omega)$ is continuous and satisfies

$$R(\omega^*) \neq 0. \tag{C28}$$

Furthermore, it is easy to see from (C27) that

$$\operatorname{sgn}\left(R(\omega^*)\right) = (-1)^{N_x(\tau^*)+1}. \tag{C29}$$

Taking derivative of (C26) and evaluating it at $\omega = \omega^*$ using Lemma 4 yield

$$\begin{aligned}\partial_\omega F(\omega^*, \tau^*) &= -R(\omega^*)(x_1'(\omega^*)\bar{x}_1^* + x_1^*\overline{x_1'(\omega^*)}) \\ &= -R(\omega^*)2\mathcal{R}(x_1'(\omega^*)\bar{x}_1^*)\end{aligned}$$

from which the conclusion follows.

If $P_M(j\omega^*, \tau^*) = 0$, then we can use (22) instead of (20), and follow a similar process. □

With Lemma 6, we can easily arrive at the following proposition.

Proposition 2. *Let Assumption IIIa be satisfied, and let $\theta(\tau)$ and $\omega(\tau)$ satisfy (C18), $\theta(\tau^*) = 2r\pi, \omega(\tau^*) = \omega^*$. Then*

$$\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_* \overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right) \neq 0. \tag{C30}$$

Furthermore,

$$\operatorname{sgn}\left(\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_* \overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right)\right) = (-1)^{N_x(\tau^*)} \operatorname{sgn}\left(\partial_\omega F(\omega^*, \tau^*)\right). \tag{C31}$$

Proof. Assume $P_M(j\omega^*, \tau^*) \neq 0$. Taking derivative of the equation

$$\Phi(\omega, \tau^*, x_1) = 0,$$

we obtain

$$\frac{\partial \Phi}{\partial \omega} + \frac{\partial \Phi}{\partial x} x_1'(\omega) = 0, \tag{C32}$$

from which we obtain

$$x_1'(\omega^*) = -\frac{\left(\frac{\partial \Phi}{\partial \omega}\right)_*}{\left(\frac{\partial \Phi}{\partial x}\right)_*} = -\frac{\left(\frac{\partial \Phi}{\partial \omega}\right)_* \overline{\left(\frac{\partial \Phi}{\partial x}\right)_*}}{\left|\left(\frac{\partial \Phi}{\partial x}\right)_*\right|^2}, \tag{C33}$$

where $(\cdot)_*$ means that the expression is evaluated at $\omega = \omega^*$, $\tau = \tau^*$, and $x_1 = x_1^* = e^{-j\omega^*\tau^*}$. From the definition of T and W, we obtain

$$\begin{aligned}\left(\frac{\partial T}{\partial \theta}\right)_* &= \left(\frac{\partial \Phi}{\partial x}\right)_* j x_1^*, \\ \left(\frac{\partial W}{\partial \omega}\right)_* &= \left(\frac{\partial \Phi}{\partial \omega}\right)_* + \left(\frac{\partial \Phi}{\partial x}\right)_* (-j\tau^* x_1^*).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_* \overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right) &= \mathcal{I}\left(\left(\frac{\partial \Phi}{\partial \omega}\right)_* \overline{\left(\frac{\partial \Phi}{\partial x}\right)_* j x_1^*}\right) \\ &= -\mathcal{R}\left(\left(\frac{\partial \Phi}{\partial \omega}\right)_* \overline{\left(\frac{\partial \Phi}{\partial x}\right)_* x_1^*}\right) \\ &= \left|\left(\frac{\partial \Phi}{\partial x}\right)_*\right|^2 \mathcal{R}(x_1'(\omega^*)\overline{x_1^*}).\end{aligned}$$

In the last step above, the equation (C33) was used. The conclusion follows from the above equation in view of Lemma 6.

If $P_M(j\omega^*, \tau^*) = 0$, then $P_0(j\omega^*, \tau^*) \neq 0$. We may use Ψ instead of Φ and follow a similar process. □

Appendix D Proof of Theorem 2

We first prove the following Lemma.

Lemma 7. *Let Assumption IIIa be satisfied. Let $\omega(\tau)$, $\theta(\tau)$ and $\lambda(\tau)$ be defined as in Theorem 2. If*

$$\frac{d^k \theta(\tau^*)}{d\tau^k} = 0, \quad k = 1, 2, \ldots, l-1, \tag{D34}$$

then

$$\mathcal{R}\left(\frac{d^k \lambda(\tau^*)}{d\tau^k}\right) = 0, \tag{D35}$$

$$\mathcal{I}\left(\frac{d^k \lambda(\tau^*)}{d\tau^k}\right) = \frac{d^k \omega(\tau^*)}{d\tau^k}, \quad k = 1, 2, \ldots, l-1, \tag{D36}$$

and

$$\mathcal{R}\left(\frac{d^l \lambda(\tau^*)}{d\tau^l}\right) = \frac{\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_* \overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right)}{\left|\left(\frac{\partial D}{\partial \lambda}\right)_*\right|^2} \left(\frac{d^l \theta(\tau^*)}{d\tau^l}\right), \tag{D37}$$

where $(\cdot)_*$ means that the expression is evaluated at $\tau = \tau^*$, $\lambda = j\omega^*$, $\omega = \omega^*$ and $\theta = 2r\pi$.

Proof. Assumption IIIa and Lemma 3 imply that

$$\partial_\lambda D(\lambda, \tau^*)|_{\lambda = j\omega^*} \neq 0. \tag{D38}$$

Therefore, the equation

$$D(\sigma + jw, \tau) = 0 \tag{D39}$$

determines continuous functions $\sigma(\tau) = \mathcal{R}(\lambda(\tau))$ and $w(\tau) = \mathcal{I}(\lambda(\tau))$ differentiable up to Nth order in a small neighborhood of 0 and ω^*, respectively, for τ in a small

neighborhood of τ^*. Define

$$S(\sigma, w, \tau) = D(\sigma + jw, \tau), \tag{D40}$$

then $\sigma(\tau)$ and $w(\tau)$ are determined by

$$S(\sigma, w, \tau) = 0. \tag{D41}$$

Taking the kth order derivative and applying Corollary 3, we obtain

$$\begin{aligned}\left(\frac{d^k S}{d\tau^k}\right)_* &= \left(\frac{\partial S}{\partial \sigma}\right)_* \left(\frac{d^k \sigma}{d\tau^k}\right)_* + \left(\frac{\partial W}{\partial \omega}\right)_* \left(\frac{d^k w}{d\tau^k}\right)_* + (U_{Sk})_* \\ &\quad + V_{Wk}|_{\omega=\omega^*, \tau=\tau^*, \omega^{(i)}(\tau^*)=w^{(i)}(\tau^*), i=1,2,\ldots,k-1} \\ &= 0,\end{aligned} \tag{D42}$$

where $U_{Sk} = U_{Sk}(\sigma, w, \tau; \sigma', \sigma'', \ldots, \sigma^{(k-1)})$ is a polynomial of $\sigma'(\tau), \sigma''(\tau), \ldots, \sigma^{(k-1)}$ that satisfies

$$U_{Sk}(\sigma, w, \tau; 0, 0, \ldots, 0) = 0. \tag{D43}$$

In arriving at (D42), we have used the fact that

$$S(\sigma, w, \tau)|_{\sigma=0} = W(\omega, \tau)|_{\omega=w}.$$

By the assumption (D34), (C20) is satisfied for $k = l$. Therefore, (C21) is satisfied for $k = 1, 2, \ldots, l$. Apply (D34) and (C21) in (C19), we obtain

$$\left(\frac{\partial W}{\partial \omega}\right)_* \left(\frac{d^k \omega}{d\tau^k}\right)_* + (V_{Wk})_* = 0, \quad k = 1, 2, \ldots, l-1. \tag{D44}$$

Using (D43) and (D44), it can be easily verified that

$$\left(\frac{d^k \sigma}{d\tau^k}\right)_* = 0, \tag{D45}$$

$$\left(\frac{d^k w}{d\tau^k}\right)_* = \left(\frac{d^k \omega}{d\tau^k}\right)_*, \ k = 1, 2, \ldots, l-1, \tag{D46}$$

satisfy the equation (D42) for $k = 1, 2, \ldots, l-1$. For $k = l$, the third term of (D42) still vanish due to (D43). Furthermore, the last term satisfies

$$V_{Wl}|_{\omega=\omega^*, \tau=\tau^*, \omega^{(i)}(\tau^*)=w^{(i)}(\tau^*), i=1,2,\ldots,l-1} = (V_{Wl})_*$$

due to (D46). Therefore,

$$\left(\frac{\partial S}{\partial \sigma}\right)_* \left(\frac{d^l \sigma}{d\tau^l}\right)_* + \left(\frac{\partial W}{\partial \omega}\right)_* \left(\frac{d^l w}{d\tau^l}\right)_* + (V_{Wl})_* = 0. \tag{D47}$$

The above can be solved for $\left(\frac{d^l\sigma}{d\tau^l}\right)_*$ by multiplying $\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}$ and taking the imaginary part,

$$\left(\frac{d^l\sigma}{d\tau^l}\right)_* = -\frac{\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}(V_{Wl})_*\right)}{\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}\left(\frac{\partial S}{\partial \sigma}\right)_*\right)}. \tag{D48}$$

But by definition,

$$\left(\frac{\partial S}{\partial \sigma}\right)_* = \left(\frac{\partial D}{\partial \lambda}\right)_*, \tag{D49}$$

$$\left(\frac{\partial W}{\partial \omega}\right)_* = j\left(\frac{\partial D}{\partial \lambda}\right)_*. \tag{D50}$$

Therefore,

$$\left(\frac{d^l\sigma}{d\tau^l}\right)_* = \frac{\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}(V_{Wl})_*\right)}{\left|\left(\frac{\partial D}{\partial \lambda}\right)_*\right|^2}. \tag{D51}$$

The equation (D37) can be obtained by comparing (C22) and (D51), and the fact that

$$\mathcal{I}\left(\overline{\left(\frac{\partial W}{\partial \omega}\right)_*}\left(\frac{\partial T}{\partial \theta}\right)_*\right) = -\mathcal{I}\left(\left(\frac{\partial W}{\partial \omega}\right)_*\overline{\left(\frac{\partial T}{\partial \theta}\right)_*}\right). \tag{D52}$$

□

With the above lemma, Theorem 2 becomes obvious.

Proof of Theorem 2. Apply Lemma 7 for $l = m$, then the equation (74) is satisfied in view of the equation (D35). To show (75), apply (C31) in Proposition 2 to (D37) in Lemma 7.

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