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# DESIGN OF QUASIPOLYNOMIAL-BASED CONTROLLERS WITH DYNAMICAL PARAMETERS - APPLICATION TO ACTIVE VIBRATION DAMPING

SAMI TLIBA, ISLAM BOUSSAADA, SILVIU-IULIAN NICULESCU, AND RICARDO FALCÓN-PRADO

ABSTRACT. Recent works on delay systems have seen the emergence of a new control structure based on delayed actions, including an auto-regressive term carrying on the control variable, with the purpose of achieving a partial pole placement in closed-loop for linear time-invariant systems of finite dimension. The ensuing controllers, called *quasipolynomial-based controllers*, correspond to some output feedback control laws with constant parameters. This specific feature has a limited number of degrees of freedom, limiting the potential performances in closed-loop. To overcome this issue, the original quasipolynomial-based controller is enhanced by using *dynamical parameters*, allowing to increase the number of degrees of freedom available for the controller's design. It turns out that the use of dynamical parameters corresponds to linear filtered terms in the control law of the original one. To emphasize the benefits brought by such a controller, the active vibration damping problem is addressed for a flexible mechanical structure equipped with a collocated pair of piezoelectric sensor and actuator.

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*Key words and phrases.* Time-delay systems, Frequency domain, Control design, MID property, Active vibration damping.

## 1. INTRODUCTION

A common feature in modeling transport and propagation phenomena and processes is the time heterogeneity that can be described by using *delays* in their mathematical representation. Such delays may be constant or time-varying, distributed (or not) over some appropriate time-intervals, depending (or not) on the state-vector. There are several ways to represent delays in the system's dynamics and, in the sequel, we are focusing on dynamical systems described by delay-differential equations (DDEs; for more insights, see, e.g., [13, 22]).

In the context of mechanical engineering problems, the effect induced by the presence of time-delays on system's dynamics was emphasized in [27] where practical applications were studied, such as the machine tools or robotic systems. For further examples, the reader is referred to [23, 12, 16] and the references therein. Furthermore, delays are intrinsically present in practical control systems. Inspired by Hazen's theory of servomechanisms<sup>1</sup> published in the 30s, one of the first approaches to handle second- and third-order systems with delay in the input was proposed by [11], [14]. For a historical perspective in the analysis and control of delay systems, we refer to [17], [27], [22].

At the end of the 1970s, the use of the delays in the controller design was introduced in [28] where the authors showed that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus it can replace the classical proportional-derivative (PD) controller.

While the pole placement represents a classical well-known control method for finite-dimensional systems, its extension to infinite dimensional systems is far to be well developed and understood. More precisely, several pole placement paradigms exist for time-delay systems, each of them has its own advantages and drawbacks, see for instance, [24, 19, 21, 10]. In particular, a recently defined paradigm, called *Partial Pole Placement* (PPP), has shown its effectiveness with respect to the robustness consideration as well as the simplicity of the resulting controller structure. The PPP paradigm is mainly based on two properties called respectively *Multiplicity-induced-dominancy* (MID) and *Coexistent-real-roots-induced-dominancy* (CRRID). As a matter of fact, the MID (respectively the CRRID) property consists of the conditions under which a given multiple zero (respectively a number of real simple zeros) of a quasipolynomial is/are dominant. For instance, in the generic quasipolynomial case, the real root of maximal multiplicity is necessarily the dominant (GMID). However, multiple roots with intermediate admissible multiplicities may be dominant or not. Thanks to this property, a consistent control strategy is proposed in [6, 1, 5], which consists of assigning a root with an intermediate admissible multiplicity once appropriate conditions guaranteeing its dominance are established. Furthermore, the MID property may be used to tune standard controllers. For instance, in [18], it is applied to the systematic tuning of the stabilizing PID controller of a first-order plant.

The contribution of this work is twofold. First, a new control strategy is proposed. Such a strategy uses the quasipolynomial-based (QPB) controller with dynamic parameters and

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<sup>1</sup>more precisely, position control systems

an appropriate tuning method for these parameters. To the best of the authors' knowledge, such a control strategy represents a novelty in the open literature. Second, the effectiveness of the proposed methodology is shown in a practical case study - the active vibration damping of a piezo-actuated flexible structure.

The remaining of the chapter is as follows: Some prerequisites, preliminaries as well as the control problem statement are briefly presented in Section 2. Section 3 includes the explicit construction of the dynamical QBP controller and an appropriate control algorithm. Next, a practical application on active vibration damping is discussed in Section 4 where the main partial derivative equations governing the considered structure are exposed and some conspicuous results are proposed. Some concluding remarks end this chapter.

*Notations.* Throughout this chapter, the following notations are used:  $\mathbb{N}^*$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  denote the sets of positive integers, real numbers, and complex numbers, respectively, and we set  $\mathbb{N} = \mathbb{N}^* \cup \{0\}$ . For a complex number  $\lambda$ ,  $\Re(\lambda)$  and  $\Im(\lambda)$  denote its real and imaginary parts, respectively. The open left and right complex half-planes are the sets  $\mathbb{C}_-$  and  $\mathbb{C}_+$ , respectively, defined by  $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$  and  $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$ . Finally,  $\deg(P)$  (resp.  $\deg(\Delta)$ ) denotes the degree of the polynomial (quasipolynomial)  $P$  (resp.  $\Delta$ ).

## 2. PREREQUISITES AND PROBLEM STATEMENT

Consider the dynamical system described by the delay-differential equation (DDE):

$$(1) \quad y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \sum_{k=0}^m \alpha_k y^{(k)}(t - \tau) = 0,$$

under appropriate initial conditions, where  $y(\cdot)$  is the real-valued unknown function,  $\tau > 0$  is the delay, and  $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_m$  are real coefficients. When the highest order of derivation appears only in the non-delayed term  $y^{(n)}(t)$ , the DDE (1) is said to be of *retarded type* if  $m < n$ , or of *neutral type* if  $m = n$ . We refer to [27, 13, 22] for a deeper discussion on DDEs and related results and properties. The characteristic function associated to (1) is the quasipolynomial  $\Delta : \mathbb{C} \mapsto \mathbb{C}$  defined by

$$(2) \quad \Delta(s) := P_0(s) + P_\tau(s) e^{-\tau s},$$

where  $P_0$  and  $P_\tau$  are the polynomials with real coefficients given by

$$(3) \quad P_0(s) = s^n + \sum_{k=0}^{n-1} a_k s^k, \quad P_\tau(s) = \sum_{k=0}^m \alpha_k s^k,$$

and the degree of  $\Delta$  is the integer  $\deg(\Delta) := n + m + 1$ . We say that a characteristic root  $s_0$  of  $\Delta$  satisfies the *MID property* if **(i)** its *algebraic multiplicity* (denoted by  $M(s_0)$ ) is *larger than one*, and **(ii)** it is *dominant* in the sense that all the characteristic roots  $\lambda_\sigma$  of the spectrum are located to the left<sup>2</sup> of  $s_0$  in  $\mathbb{C}_-$ . In other words,  $s_0$  is the rightmost root of the spectrum and defines the *spectral abscissa* of the quasipolynomial  $\Delta$ . In the case  $M(s_0) = \deg(\Delta)$ , it was shown in [20] (case  $m = n - 1$ ) and [4] (general case  $m \leq n$ ) that  $s_0$  satisfies the MID property. This "limit" case is also called *generic MID* or GMID for short.

**Remark 2.1.** As noticed in [4], the GMID does not allow any degree of freedom in assigning  $s_0$ . In order to allow for some additional degrees of freedom when assigning  $s_0$ , one can relax such a constraint by forcing the root  $s_0$  to have a multiplicity lower than the maximal one, and consider, for instance, the delay as a *free tuning parameter*.

<sup>2</sup>In other words,  $\lambda_\sigma$  satisfies the condition  $\Re(\lambda_\sigma) \leq \Re(s_0)$ .

**2.1. Problem statement.** Consider a linear time-invariant (LTI) system  $\mathcal{S}$  with a scalar control input  $u(t)$ , a scalar measured output  $y(t)$ , a scalar disturbance input  $w(t)$  and an output of interest  $z(t)$ . The model of  $\mathcal{S}$  based on transfer functions is given by

$$(4) \quad \mathcal{S} \begin{cases} Z(s) = \frac{N_{wz}(s)}{\psi(s)} W(s) + \frac{N_{uz}(s)}{\psi(s)} U(s), \\ Y(s) = \frac{N_{wy}(s)}{\psi(s)} W(s) + \frac{N_{uy}(s)}{\psi(s)} U(s), \end{cases}$$

where the polynomials, with real coefficients, have the form:

$$(5) \quad \begin{aligned} N_{wz}(s) &:= \sum_{k=0}^{n_p} n_{wz_k} s^k, & N_{uz}(s) &:= \sum_{k=0}^{n_p} n_{uz_k} s^k, \\ N_{wy}(s) &:= \sum_{k=0}^{n_p} n_{wy_k} s^k, & N_{uy}(s) &:= \sum_{k=0}^{n_p} n_{uy_k} s^k \\ && \text{and } \psi(s) &:= s^{n_p} + \sum_{k=0}^{n_p-1} a_k s^k, \end{aligned}$$

where  $n_p$  is the order of the system. For the sake of simplicity,  $\psi(\cdot)$  is chosen to be a monic polynomial. The control model, given by  $\frac{N_{uy}(s)}{\psi(s)}$ , is assumed to be in its minimal form, such that  $N_{uy}(\cdot)$  and  $\psi(\cdot)$  are co-prime polynomials. This assumption simply means that the  $n_p$  poles of the linear systems are not simplified by the roots of  $N_{uy}(\cdot)$ . In practice, it means that the dynamics related to these poles are controllable and observable.

The proposed control problem is to *design an output feedback controller in order to assign the rightmost root of the closed-loop system on a desired location in the open left-half part of the complex plane.*

**2.2. MID property and QPB controller design.** To solve the control problem above, we consider *low-complexity controllers* based on quasipolynomials. Such controllers are called *QPB controllers*, and were introduced in [32].

**Definition 2.2.** Let  $n_0, n_{\tau_0}, d_0, d_{\tau_0} \in \mathbb{R}$  be such that  $\tau \in \mathbb{R}_+$ ,  $d_0 \neq 0$  and at least one of the two other numbers  $n_0$  and  $n_{\tau_0}$  is nonzero. Then, a generic output feedback *QPB controller* is defined by the following continuous-time delay-difference equation:

$$(6) \quad u(t) = -\frac{d_{\tau_0}}{d_0} u(t - \tau) + \frac{n_0}{d_0} y(t) + \frac{n_{\tau_0}}{d_0} y(t - \tau).$$

In the Laplace domain, (6) yields  $U(s) = C(s, \tau) Y(s)$ , with

$$(7) \quad C(s, \tau) := \frac{n_0 + n_{\tau_0} e^{-\tau s}}{d_0 + d_{\tau_0} e^{-\tau s}}.$$

This control law is based on nothing else than an addition of proportional and delayed-proportional terms carrying on the signals  $u(t)$  and  $y(t)$ , which makes it having a *low complexity* feature. As indicated above, the parameters of the controller are the four scalars  $n_0, n_{\tau_0}, d_0, d_{\tau_0}$ , and the positive time delay  $\tau$ , giving an amount of 4 independent degrees-of-freedom<sup>3</sup> for the pole assignment problem subject to the constraint that the controller is well-posed. It should be mentioned that the delay  $\tau$  is considered here as a *design parameter* of the QPB controller, used with the other gains to assign the rightmost root of the closed-loop system.

<sup>3</sup>Without any loss of generality, one may assume  $d_0 = 1$ .

Consider now the closed-loop characteristic function  $\Delta(s)$  of the system  $\mathcal{S}$  with the standard QPB controller  $C(s, \tau)$ . This latter is written as in (2) where here,  $P_0(s) := d_0 \psi(s) - n_0 N_{uy}(s)$  and  $P_\tau(s) := d_{\tau_0} \psi(s) - n_{\tau_0} N_{uy}(s)$ .

The underlying idea can be resumed as follows: the QPB controller is designed to assign the closed-loop rightmost root by using the multiple-induced-dominancy (MID) property introduced and shortly presented in the previous section. The main result on the MID property is recalled next in order to describe the design method leading to the sought gains  $n_0, n_{\tau_0}, d_0$  and  $d_{\tau_0}$ , as well as the delay  $\tau$  used as a design parameter, in order to achieve the assignment of the rightmost root  $s_0 \in \mathbb{C}_-$  while guaranteeing the closed-loop stability.

**2.3. MID-based partial pole placement.** The partial pole placement used throughout this chapter is based on a control-oriented MID property as introduced and discussed in [5], see also [6, 1]. Let us first consider the generic quasipolynomial  $\Delta(s)$  in (2) with  $m \leq n$ . The control-oriented MID property's main idea consists in forcing a given negative scalar  $s_0$  to be a multiple spectral root of the system's closed-loop characteristic function given by  $\Delta(s)$ , and leading to some algebraic relations among the controller's parameters. More precisely, when the assigned root reaches a multiplicity at least equal to  $n$ , this guarantees some integral representation of the corresponding quasipolynomial as emphasized in [9]. Next, the controller's parameters are obtained thanks to the parametric conditions reflecting the dominant feature of the multiple spectral root, see for instance [6, 1]. The following Theorem from [5] gives explicitly the integral representation of the quasipolynomial.

**Theorem 2.3.** *Let  $\tau > 0$ ,  $s_0 \in \mathbb{R}$ , and consider the quasipolynomial  $\Delta$  from (2)–(3). The number  $s_0$  is a root of  $\Delta$  with multiplicity at least  $n + m$  if, and only if there exists  $A \in \mathbb{R}$  such that*

$$(8) \quad \Delta(s) = \frac{\tau^m (s - s_0)^{n+m}}{(m-1)!} \int_0^1 t^{m-1} (1-t)^{n-1} (1-At) e^{-t\tau(s-s_0)} dt.$$

A useful technique is to establish *a priori* information on the location of roots of  $\Delta$  with real part greater than  $s_0$ , and in particular bounds on their imaginary parts.

A standard first step to do so is to introduce the normalized quasipolynomial  $\tilde{\Delta}(\lambda) = \tau^n \Delta(s_0 + \frac{\lambda}{\tau})$ , which can be written as  $\tilde{\Delta}(\lambda) = \tilde{P}_0(\lambda) + e^{-\lambda} \tilde{P}_\tau(\lambda)$  for some suitable polynomials  $\tilde{P}_0$  and  $\tilde{P}_\tau$  of degrees  $n$  and  $m$ , respectively. Hence, the problem of studying eventual roots of  $\Delta$  with real part greater than  $s_0$  reduces to the study of eventual roots of  $\tilde{\Delta}$  with positive real part.

A possible strategy to do so is to notice that any root  $\lambda$  of  $\tilde{\Delta}$  satisfies

$$|\tilde{P}_0(x + i\omega)|^2 e^{2x} = |\tilde{P}_\tau(x + i\omega)|^2,$$

where  $x = \Re(\lambda)$  and  $\omega = \Im(\lambda)$ . In particular, if  $\lambda$  has nonnegative real part, then  $e^{2x} \geq T_\ell(x)$ , where, for  $\ell \in \mathbb{N}$ , the polynomial  $T_\ell$  is the truncation of the Taylor expansion of  $e^{2x}$  at order  $\ell$ , i.e.,  $T_\ell(x) = \sum_{k=0}^{\ell} \frac{(2x)^k}{k!}$ . Hence, any root  $\lambda = x + i\omega$  of  $\tilde{\Delta}$  with nonnegative real part satisfies

$$\mathcal{F}(x, \omega) \geq 0,$$

where  $\mathcal{F}$  is the polynomial given by

$$\mathcal{F}(x, \omega) = |\tilde{P}_\tau(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 T_\ell(x).$$

In addition,  $\mathcal{F}$  only depends on  $\omega$  through  $\omega^2$  (which is a consequence of the fact that  $\tilde{P}_0$  and  $\tilde{P}_\tau$  are polynomials with real coefficients), and one may thus introduce the variable

$\Omega := \omega^2$  and define the polynomial  $H$  by setting  $H(x, \Omega) = F(x, \sqrt{\Omega})$  for  $\Omega \geq 0$ . Hence, any root  $\lambda = x + i\omega$  of  $\tilde{\Delta}$  with nonnegative real part satisfies

$$(9) \quad H(x, \Omega) \geq 0.$$

One can thus establish a bound on the imaginary parts for the roots of  $\tilde{\Delta}$  by exploiting the polynomial inequality (9). This has been done for some low-order cases in [2, 3]. In particular, all these works have shown that it is sufficient to bound the absolute value of the imaginary parts of the roots in the right half-plane by  $\pi$ , as one can in general easily exclude by other arguments [4] the possibility of having roots in the right-half plane with imaginary part at most  $\pi$ , thus concluding the proof for the dominance of  $s_0$ .

The procedure described in this subsection is synthesized in Algorithm 1 (see [2]), in which one increases the order of the Taylor expansion of  $e^{2x}$  until a suitable bound is found.

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**Algorithm 1:** Estimation of a frequency bound for time-delay differential equations with a single delay

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**Input:**  $\tilde{\Delta}(\lambda) = \tilde{P}_0(\lambda) + \tilde{P}_\tau(\lambda) e^{-\lambda}$ ; // Normalized quasipolynomial  
**Input:**  $\text{maxOrd}$ ; // Maximal order  
// Initialization  
 $\text{ord} = 0$ ; //  $\text{ord}$ : order of truncation of the Taylor expansion of  $e^{2x}$ ;  
 $\text{dominance} = \text{false}$ ;  
**while** (*not dominance*) and ( $\text{ord} \leq \text{maxOrd}$ ) **do**  
    Set  $\mathcal{F}(x, \omega) = |\tilde{P}_\tau(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 T_{\text{ord}}(x)$ ;  
    //  $T_{\text{ord}}(x)$ : Taylor expansion of  $e^{2x}$  of order =  $\text{ord}$   
    Set  $H(x, \Omega) = \mathcal{F}(x, \sqrt{\Omega})$ ; //  $H$  is a polynomial  
    Set  $\Omega_k(x)$  as the  $k$ -th real root of  $H(x, \cdot)$ ;  
    **if**  $\sup_{x \geq 0} \max_k \Omega_k(x) \leq \pi^2$  **then**  
         $\perp$   $\text{dominance} = \text{true}$ ;  
     $\text{ord} = \text{ord} + 1$ ;  
**Output:** Frequency bound: If dominance is true, then  $|\omega| \leq \pi$  for every root of  $\tilde{\Delta}$  with positive real part;

---

### 3. MAIN RESULTS

**3.1. Dynamical QPB controller.** Based on the standard QPB controller's structure, recalled in Section 2, we propose to extend the features of such a controller by introducing some dynamical parameters instead of the static ones  $n_0, n_{\tau_0}, d_0$  and  $d_{\tau_0}$ . The aim is to offer more degrees of freedom to handle high-order linear systems, in order to cope with various issues such as the robustness one, arising for example with the spillover phenomenon in the control of flexible structures.

**Definition 3.1.** The output feedback QPB controller with dynamical parameters is defined, in the Laplace domain, by

$$(10) \quad D(s, \tau) := \frac{N_0(s) + N_{\tau_0}(s) e^{-\tau s}}{D_0(s) + D_{\tau_0}(s) e^{-\tau s}},$$

where  $N_0(s)$ ,  $N_{\tau_0}(s)$ ,  $D_0(s)$ ,  $D_{\tau_0}(s)$  are polynomials in  $s$  with finite degree. The total amount of available independent parameters, denoted  $N_P$ , is given by  $N_P := \deg(N_0) + \deg(N_{\tau_0}) + \deg(D_0) + \deg(D_{\tau_0}) + 4$ .

**Remark 3.2.** It is important to note that the degrees of these polynomials are assumed to be such that all the following transfer functions remain proper for practical purposes:

$$F_y(s) := \frac{N_0(s)}{D_0(s)}, F_{y_d}(s) := \frac{N_{\tau_0}(s)}{D_0(s)} \text{ and } F_{u_d}(s) := \frac{D_{\tau_0}(s)}{D_0(s)}.$$

**Fact 3.3.** *The closed-loop system  $\mathcal{S}$  in (4) with the Dynamical QPB controller in (10), has the same characteristic equation than in (2) where now,  $P_0(s) := D_0(s)\psi(s) - N_0(s)N_{uy}(s)$  and  $P_\tau(s) := D_{\tau_0}(s)\psi(s) - N_{\tau_0}(s)N_{uy}(s)$ . Moreover,  $\deg(\Delta) = \deg(D_0) + \deg(D_{\tau_0}) + 2n_p + 1$ .*

This fact shows that the MID property used for the design of the standard QPB controller can also be used for the dynamical case. The main difference relies on the practical implementation of the controller. We shall say few words about that in the next subsection.

**3.2. Some practical implementation schemes.** Let us denote  $f_{u_d}(t)$ ,  $f_y(t)$  and  $f_{y_d}(t)$  the inverse Laplace transform of  $F_{u_d}(s)$ ,  $F_y(s)$  and  $F_{y_d}(s)$  respectively. In time domain, the control law derived from (10) reads:

$$(11) \quad u(t) := -f_{u_d}(t) * u(t - \tau) + f_y(t) * y(t) + f_{y_d}(t) * y(t - \tau),$$

where the symbol  $*$  stands for the time domain convolution product of causal signals.

**Remark 3.4.** It is worth noticing that the control signal (11) is, with the extended QPB controller, the result of filtered terms carrying on the delayed control signal and the measured output as well as its delayed part, that are all added. In consequence, the complexity of the control law is slightly increased w.r.t. the one from the standard QPB controller, but with the benefit of a greater set of available degrees-of-freedom, *i.e.* the coefficients of the polynomials introduced in Def. 3.1.

**Remark 3.5.** Note that the filters  $F_{u_d}(s)$ ,  $F_y(s)$  and  $F_{y_d}(s)$  share the same poles. It can also be interesting to filter each term of the control law (11) with a separate filter, *ie* each one with its own dynamic.

Let us define the following proper linear transfer functions  $G_{u_d}(s) := \frac{N_{u_d}(s)}{D_{u_d}(s)}$ ,  $G_y(s) := \frac{N_y(s)}{D_y(s)}$ ,  $G_{y_d}(s) := \frac{N_{y_d}(s)}{D_{y_d}(s)}$ . Those transfer functions can be considered as the mathematical models of distinct linear filters. The QPB controller with dynamical parameters is defined by the following control law in the time domain

$$(12) \quad u(t) = -g_{u_d}(s) * u(t - \tau) + g_y(t) * y(t) + g_{y_d}(t) * y(t - \tau)$$

where  $g_{u_d}(t)$ ,  $g_y(t)$  and  $g_{y_d}(t)$  are the inverse Laplace transforms of the previous transfer functions, *ie* their associated impulse responses. The resulting controller derived from this control law expressed in the Laplace domain leads to

$$(13) \quad U(s) = D(s, \tau)Y(s),$$

$$(14) \quad \text{where } D(s, \tau) := \frac{G_y(s) + G_{y_d}(s)e^{-\tau s}}{1 + G_{u_d}(s)e^{-\tau s}}.$$

This last corresponds to a QPB controller with dynamical parameters as in (10), with  $N_0(s) := D_{u_d}(s)N_y(s)D_{y_d}(s)$ ,  $N_{r_0}(s) := D_{u_d}(s)N_{y_d}(s)D_y(s)$ ,  $D_0(s) := D_{y_d}(s)D_y(s)D_{u_d}(s)$  and  $D_{r_0}(s) := N_{u_d}(s)D_y(s)D_{y_d}(s)$ .

**3.3. Obtaining the parameters of the QPB controller.** The QPB controller's structure, *ie* the degree of each polynomial composing it in (11), is now assumed to be fixed to handle the considered control problem constraints. Then, thanks to the linear dependency of  $\Delta$  w.r.t. the control parameters, the procedure leading to get these parameters is based on a basic elimination procedure allowing the resolution of the equation set, stating the required multiplicity of the root  $s_0$  for  $\Delta$ . In other words, given  $M(s_0)$ , under the necessary and sufficient conditions  $N_P \geq M(s_0) \leq \deg(\Delta)$ , this procedure consists in solving sequentially the set of equations

$$(15) \quad \Delta^{(k-1)}(s) \Big|_{s=s_0} = 0,$$

for  $k = 1$  to  $M(s_0)$  in the controller's parameters, where  $\Delta^{(j)}(s)$  stands for the  $j^{\text{th}}$  derivative of  $\Delta(s)$  in terms of  $s$ .

#### 4. APPLICATION TO ACTIVE VIBRATION DAMPING

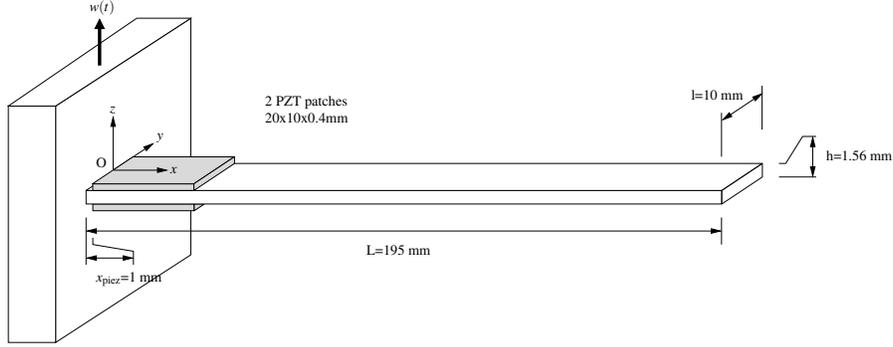
**4.1. System description and problem statement.** The previous results are now applied to the active vibration control problem presented in [29], [7], [8] and in [32]. It concerns a lightly-damped beam-like flexible structure with one clamped edge and the other free. It is an aluminum-based beam that is equipped with a piezoelectric rectangular patch used as an actuator and rigidly bonded on one side of the beam near the fixation. Another piezoelectric patch, with the same dimensions and used as a sensor, is collocated to the actuator and rigidly bonded on the other side. These piezoelectric patches are composed of a ceramic based on Lead-Zirconate-Titanate, also known as PZT in the literature. They are polarized along the thickness. Each of the two main faces of these patches are recovered by an electroconductive layer, an electrode, making an independent electrical equipotential on each of these faces. Both the sensor and the actuator are made with the same piezoelectric material and thus have the same electromechanical properties.

A sketch of this system with the main dimensions is depicted in Fig. 1. The cartesian reference used to describe the spatial variables, is depicted in detail in Fig. 1b. Once equipped with these piezoelectric transducers, the piezo-actuated flexible structure has the ability to be deformed thanks to an appropriate voltage, denoted by  $u$  and expressed in Volt, applied across the piezoelectric actuator. The piezoelectric sensor allows then to monitor the dynamical strains. It delivers an electrical voltage denoted by  $y$  and expressed in Volt. The whole is often called a *smart structure* since it can be programmed to reduce its resonant features. More about the active vibration damping using piezoelectric patches can be found in [29] or [25].

In this work, the considered smart structure is assumed to be embedded in a mobile support. The mobile support is subjected to an acceleration, denoted by  $w$  in the sequel, and it is moving along the  $z$  axis. Such a composite system obeys mechanical and electrostatic laws expressed through coupled partial derivative equations, giving place to an *infinite-dimensional system*.

Let us recall the main equations governing the dynamical motion of structural deformations and electrical variations, in the three dimensional space.

In the following, the superscript  $\langle l \rangle$  will label the variables that rule the mechanical and electrical motions carrying on the body  $\Omega_b$ ,  $\Omega_a$  or  $\Omega_s$  by specifying  $l = b$ ,  $l = a$  or  $l = s$  respectively. The main physical variables, all time and space variable dependent, are



(a) Sketch of the system with the whole dimensions.

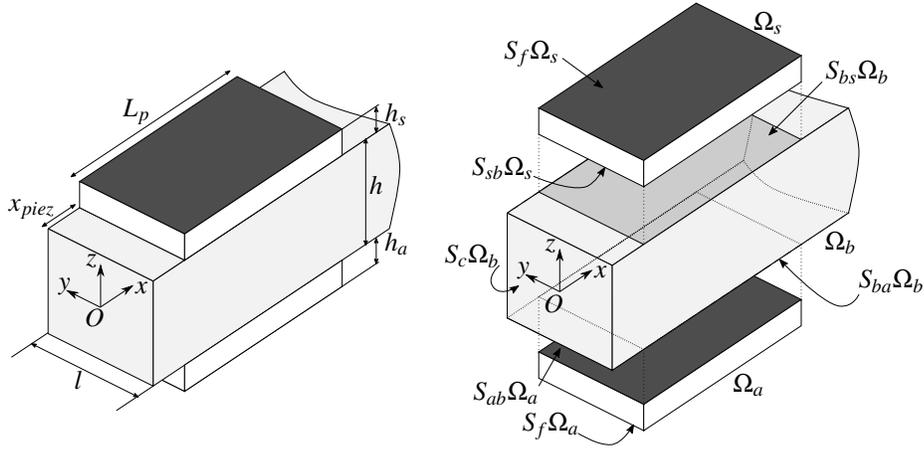

 (b) Dimensions of the piezoelectric patches:  $h = 1.56 \text{ mm}$ ,  $h_s = h_a = 0.4 \text{ mm}$ ,  $L_p = 20 \text{ mm}$ ,  $l = 10 \text{ mm}$ ,  $x_{\text{piez}} = 1 \text{ mm}$ . (c) Exploded view. The body  $\Omega_a$  is the piezoelectric actuator; The body  $\Omega_s$  is the piezoelectric sensor; The body  $\Omega_b$  is the aluminum beam;

Figure 1: The piezo-actuated beam with one clamped edge and equipped with two piezoelectric patches in ceramic made of Lead-Zirconate-Titanate (PZT).

- the stress (second-order) tensor, denoted  $\sigma^{(l)}$  for  $l = b$ ,  $l = a$  and  $l = s$ , where

$$\sigma^{(l)} = \begin{bmatrix} \sigma_{xx}^{(l)} & \sigma_{xy}^{(l)} & \sigma_{xz}^{(l)} \\ \sigma_{yx}^{(l)} & \sigma_{yy}^{(l)} & \sigma_{yz}^{(l)} \\ \sigma_{zx}^{(l)} & \sigma_{zy}^{(l)} & \sigma_{zz}^{(l)} \end{bmatrix} \in \mathbb{R}^{3 \times 3};$$

- the displacement vector field, denoted  $\mu^{(l)}$  for  $l = b$ ,  $l = a$  and  $l = s$ , where  $\mu^{(l)} = [\mu_x^{(l)} \quad \mu_y^{(l)} \quad \mu_z^{(l)}]^T \in \mathbb{R}^3$ ;
- the strain tensor, denoted  $\varepsilon^{(l)}$  for  $l = b$ ,  $l = a$  and  $l = s$ , where

$$\varepsilon^{(l)} = \begin{bmatrix} \varepsilon_{xx}^{(l)} & \varepsilon_{xy}^{(l)} & \varepsilon_{xz}^{(l)} \\ \varepsilon_{yx}^{(l)} & \varepsilon_{yy}^{(l)} & \varepsilon_{yz}^{(l)} \\ \varepsilon_{zx}^{(l)} & \varepsilon_{zy}^{(l)} & \varepsilon_{zz}^{(l)} \end{bmatrix} \in \mathbb{R}^{3 \times 3};$$

- the body force field vector, denoted  $f^{(l)}$  for  $l = b$ ,  $l = a$  and  $l = s$ , where  $f^{(l)} = \begin{bmatrix} f_x^{(l)} & f_y^{(l)} & f_z^{(l)} \end{bmatrix}^T \in \mathbb{R}^3$ ;
- the electric displacement field vector, denoted  $D^{(l)}$  for  $l = a$  and  $l = s$ , where  $D^{(l)} = \begin{bmatrix} D_x^{(l)} & D_y^{(l)} & D_z^{(l)} \end{bmatrix}^T \in \mathbb{R}^3$ ;
- the electric field vector, denoted  $E^{(l)}$  for  $l = a$  and  $l = s$ , where  $E^{(l)} = \begin{bmatrix} E_x^{(l)} & E_y^{(l)} & E_z^{(l)} \end{bmatrix}^T \in \mathbb{R}^3$  with the same consideration as before;
- the electric scalar potential, denoted  $\phi^{(l)} \in \mathbb{R}$  for  $l = a$  and  $l = s$ ;

All the vectors and tensors are expressed in the orthogonal basis  $(x, y, z)$  depicted in Fig. 1b.

The features of the piezoelectric material used for the sensor as well as for the actuator are :

- $[c^E] \in \mathbb{R}^{6 \times 6}$  is the elastic constitutive tensor for the piezoelectric bodies;
- $[e] \in \mathbb{R}^{3 \times 6}$  is the piezoelectric strain tensor;
- $[\epsilon^E] \in \mathbb{R}^{3 \times 3}$  is the dielectric permittivity tensor.

Their main expressions of the piezoelectric constitutive equations are:

$$(16) \quad [c^E] = \begin{bmatrix} c_{11}^E & c_{12}^E & c_{13}^E & 0 & 0 & 0 \\ c_{12}^E & c_{11}^E & c_{13}^E & 0 & 0 & 0 \\ c_{13}^E & c_{13}^E & c_{33}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^E & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^E \end{bmatrix},$$

$$(17) \quad [d] = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix},$$

$$(18) \quad [\epsilon^\sigma] = \begin{bmatrix} \epsilon_{11}^\sigma & 0 & 0 \\ 0 & \epsilon_{11}^\sigma & 0 \\ 0 & 0 & \epsilon_{33}^\sigma \end{bmatrix},$$

$$(19) \quad [\epsilon^E] = [\epsilon^\sigma] - [d][c^E][d]^T,$$

$$(20) \quad [e] = [d][c^E].$$

The numerical data of the corresponding parameters are reported in Table 1.

Moreover,  $\rho^{(l)} > 0$  denotes the density of the body  $\langle l \rangle$  for  $l = b$ ,  $l = a$  or  $l = s$  where  $\rho^{(a)} = \rho^{(s)}$ .

In the sequel, the partial derivative of the component  $\sigma_{ij}$  of the tensor  $\sigma \in \mathbb{R}^{3 \times 3}$  with respect to a space variable  $i$  ( $i = x, y$  or  $z$ ) will be denoted by  $\sigma_{ij,i}$ , and the partial derivative of the component  $V_j$  of a vector  $V \in \mathbb{R}^3$  with respect to a space variable  $i$  or the time variable  $t$  will be denoted by  $V_{j,i}$ ,  $V$  designating either  $E$ ,  $D$  or  $\mu$ . Finally,  $\phi_{,i}$  stands for the partial derivative of the scalar potential  $\phi$  with respect to a space variable  $i$ .

The mechanical equilibrium equations, for  $l = b$ ,  $l = a$  and  $l = s$ , are given by

$$(21) \quad \sigma_{ix,x}^{(l)} + \sigma_{iy,y}^{(l)} + \sigma_{iz,z}^{(l)} + f_i = \rho \mu_{i,tt}^{(l)}, \quad i = x, y, z,$$

$$(22) \quad \text{with } \sigma_{ij}^{(l)} = \sigma_{ji}^{(l)}, \quad i, j = x, y, z.$$

Table 1: Specific features for the piezoceramics PIC 151 of Physik Instrumente™

parameter	unit	value	parameter	unit	value
density	$kg/m^3$	7760	$N_1$	$Hz \cdot m$	1384
Q		88	$N_3$	$Hz \cdot m$	181.7
$\epsilon_{11}^\sigma$		1936	$N_5$	$Hz \cdot m$	1050
$\epsilon_{33}^\sigma$		2109	$N_p$	$Hz \cdot m$	1915
$\epsilon_{11}^\epsilon$		1110	$N_t$	$Hz \cdot m$	2118
$\epsilon_{33}^\epsilon$		852	$d_{31}$	$m/V$	$-214 \cdot 10^{-12}$
$\tan \delta$		15.7	$d_{33}$	$m/V$	$423 \cdot 10^{-12}$
$k_{31}$		0.382	$d_{15}$	$m/V$	$610 \cdot 10^{-12}$
$k_{33}$		0.697	$g_{31}$	$V m/N$	$-11.5 \cdot 10^{-3}$
$k_{15}$		0.653	$g_{33}$	$V m/N$	$21.8 \cdot 10^{-3}$
$k_p$		0.663	$g_{15}$	$V m/N$	$36.5 \cdot 10^{-3}$
$k_t$		0.528	$e_{31}$	$N/V/m$	-9.6
$\nu$		0.34	$e_{33}$	$N/V/m$	15.1
			$e_{15}$	$N/V/m$	12.0
$s_{11}^E$	$m^2/N$	$1.683 \cdot 10^{-11}$	$c_{11}^E$	$N/m^2$	$1.076 \cdot 10^{11}$
$s_{33}^E$	$m^2/N$	$1.900 \cdot 10^{-11}$	$c_{33}^E$	$N/m^2$	$1.004 \cdot 10^{11}$
$s_{55}^E$	$m^2/N$	$5.096 \cdot 10^{-11}$	$c_{55}^E$	$N/m^2$	$1.962 \cdot 10^{10}$
$s_{12}^E$	$m^2/N$	$-5.656 \cdot 10^{-12}$	$c_{12}^E$	$N/m^2$	$6.312 \cdot 10^{10}$
$s_{13}^E$	$m^2/N$	$-7.107 \cdot 10^{-12}$	$c_{13}^E$	$N/m^2$	$6.385 \cdot 10^{10}$
$s_{44}^E$	$m^2/N$	$5.096 \cdot 10^{-11}$	$c_{44}^E$	$N/m^2$	$1.962 \cdot 10^{10}$
$s_{66}^E$	$m^2/N$	$4.497 \cdot 10^{-11}$	$c_{66}^E$	$N/m^2$	$2.224 \cdot 10^{10}$
$E_{max}$	$V/mm$	2000			
$t^{\circ}_{curie}$	$^{\circ}C$	250			

Under the assumptions of small displacements and small strains, the first-order relation between the strain tensor and the displacement field's vector is

$$(23) \quad \epsilon_{ij}^{(l)} = \frac{1}{2} \left( \mu_{i,j}^{(l)} + \mu_{j,i}^{(l)} \right),$$

for  $l = b$ ,  $l = a$  and  $l = s$ . The Maxwell's equations corresponding to the electrical equilibrium within  $\Omega_a$  and  $\Omega_s$ , under the assumptions of a perfect piezoelectric material (with no electrical charge inside the material) and the quasi-static electric field's one, are

$$(24) \quad D_{x,x}^{(l)} + D_{y,y}^{(l)} + D_{z,z}^{(l)} = 0,$$

for  $l = a$  and  $l = s$ . Then, the electric field  $E^{(l)}$  is related to the electrical scalar potential  $\phi^{(l)}$  by

$$(25) \quad E_i^{(l)} = -\phi_{,i}^{(l)}.$$

Lastly, the linear constitutive equations of the piezoelectric bodies  $\Omega_a$  and  $\Omega_s$  obey to

$$(26) \quad \begin{cases} \text{Vec}\{\boldsymbol{\sigma}^{(l)}\} = [c^E] \text{Vec}\{\boldsymbol{\varepsilon}^{(l)}\} - [e]^T E^{(l)}, \\ D^{(l)} = [e] \text{Vec}\{\boldsymbol{\varepsilon}^{(l)}\} + [\epsilon^E] E^{(l)}, \end{cases}$$

with

$$(27) \quad \text{Vec}\{\boldsymbol{\sigma}^{(l)}\} := \begin{bmatrix} \sigma_{xx}^{(l)} & \sigma_{yy}^{(l)} & \sigma_{zz}^{(l)} & \sigma_{yz}^{(l)} & \sigma_{xz}^{(l)} & \sigma_{xy}^{(l)} \end{bmatrix}^T,$$

$$(28) \quad \text{Vec}\{\boldsymbol{\varepsilon}^{(l)}\} := \begin{bmatrix} \varepsilon_{xx}^{(l)} & \varepsilon_{yy}^{(l)} & \varepsilon_{zz}^{(l)} & 2\varepsilon_{yz}^{(l)} & 2\varepsilon_{xz}^{(l)} & 2\varepsilon_{xy}^{(l)} \end{bmatrix}^T.$$

The one of the aluminum-based beam is given by

$$(29) \quad \boldsymbol{\sigma}^{(b)} = \frac{Y}{1+\nu} \boldsymbol{\varepsilon}^{(b)} + \frac{\nu Y}{(1+\nu)(1-2\nu)} \text{Tr}\{\boldsymbol{\varepsilon}^{(b)}\} \mathbb{1}_3,$$

where  $\mathbb{1}_3$  is the  $3 \times 3$  identity matrix,  $\text{Tr}$  is the matrix trace operator,  $Y$  and  $\nu$  are the Young's modulus and Poisson's coefficient characterizing the constitutive law of the used aluminum. There values are  $Y = 78 \cdot 10^9$  (Pa) and  $\nu = 0.3$ .

To complete the problem statement, the boundary conditions are specified on the different surfaces indicated in Fig. 1c. The Dirichlet type conditions are:

- $\mu = 0$  on  $S_c \Omega_b$ , expressing a zero displacement of clamped edge;
- $\phi = 0$  on  $S_{sb} \Omega_s \cup S_{ab} \Omega_a$ , expressing the ground voltage on the bounded faces of the piezoelectric patches;  
Note that the perfect gluing of the piezoelectric patches on the aluminum beam is expressed by  $S_{sb} \Omega_s = S_{bs} \Omega_b$  and  $S_{ab} \Omega_a = S_{ba} \Omega_b$ .
- $\phi = u$  on  $S_f \Omega_a$ , expressing an electrical equipotential on the free face of the piezoelectric actuator that is the controlled variable;
- $\phi = y$  on  $S_f \Omega_s$ , expressing an electrical equipotential on the free face of the piezoelectric sensor that is the measured variable;

The only Neumann condition carries on the sensor's free electrode, where there is no prescribed surface electric charge since it is assumed that the sensor is electrically insulated so that the processing of the sensor's measurement  $y$  does not interfere with the produced electric charges. This is expressed as

$$D_x n_x + D_y n_y + D_z n_z = 0,$$

where  $n_x$ ,  $n_y$  and  $n_z$  are the unit vectors orthogonal to the free faces (boundaries) of the body  $\Omega_s$ .

The whole system is assumed to be at rest at  $t = 0$ . Hence, the initial conditions are all zero, say  $\mu_i = 0$  and  $\mu_{i,t} = 0$ ,  $\phi = 0$  and  $\phi_{,t} = 0$  on  $\Omega_b \cup \Omega_s \cup \Omega_a$ , for  $i = x, y, z$ .

These coupled partial derivative equations are difficult to handle analytically. Moreover, a finite-dimensional inputs-to-outputs model in transfer function form is desired to suit our control method. To this end, the next step of the modeling is the approximation of the solution thanks to the finite element method where a deep presentation can be found in [15, 26]. This framework is very well suited to take into account the electromechanical

coupling between piezoelectric patches and thin mechanical structures, as emphasized in [31].

The inputs-to-outputs dynamical model of finite dimension is obtained thanks to a *finite element modeling*, whose presentation is out of the scope of this work. More details are given in [8] and references therein. Fig. 2 shows both the meshing and the type of elements used to approximate the infinite-dimensional problem presented previously.

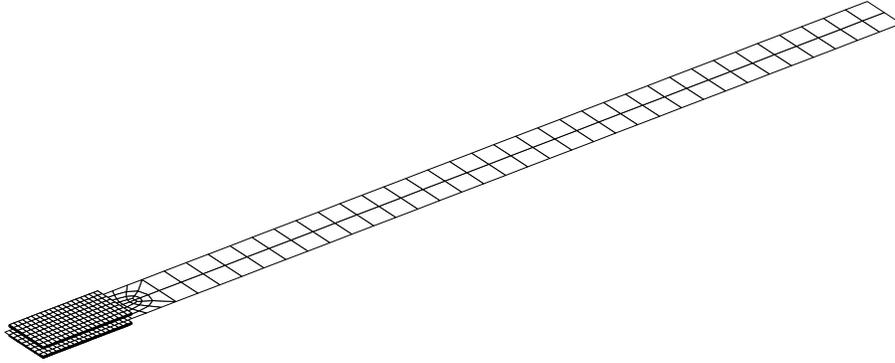


Figure 2: Meshing of the composite structure for the finite element modeling.

The obtained finite-dimensional model is linear and of order 24, which is enough for an accurate description in the low-frequency range, up to 3500 ( $Hz$ ). For control purposes, a design model is derived from the previous full-order one after a reduction to order 2, containing only the first mode dynamic and preserving the static inputs-to-outputs gains. The numerical data of the reduced order model are given in [8]. The frequency responses of both models are presented in Fig. 3. These models are given with the form (4), where  $Z(s)$  stands here for the Laplace transform of the relative acceleration of the point located at the middle-end of the beam (relatively to the clamped edge's acceleration),  $Y(s)$  corresponds to the voltage across the piezoelectric sensor  $y$ ,  $U(s)$  is the voltage applied across the piezoelectric actuator  $u$  and  $W(s)$  is the acceleration imposed to the clamped edge of the beam  $w$ .

The flexible beam is submitted to a shock-like disturbance  $w$ , applied to its clamped edge. It is represented by a rectangular signal of magnitude 1 ( $m/s^2$ ) with a pulse-width of  $6 \cdot 10^{-3}$  ( $s$ ). In response to this disturbance signal, the control objective consists in damping the dominant vibration mode without degrading the natural damping of the other modes located at higher frequencies. The first three vibration modes of bending type are presented in Fig. 4. The first one has a resonant frequency at 38.8  $Hz$ .

**4.2. Controller design.** The *disturbance rejection problem* considered here is formulated as a *robust performance control problem*, where the controller has to be robust w.r.t. the vibration modes neglected in the design model. To cope with this issue, designers frequently introduce a high-order roll-off filter in serial with the controller, with a cutoff frequency located between the last mode's frequency included in the design model and the first neglected mode's one. The choice of this filter's parameters is generally rather empirical. The fundamental result behind the use of this filter is the *low-gain theorem* recalled in [33] pp. 204. Based on the same idea, in this work, the choice is made to use a first-order

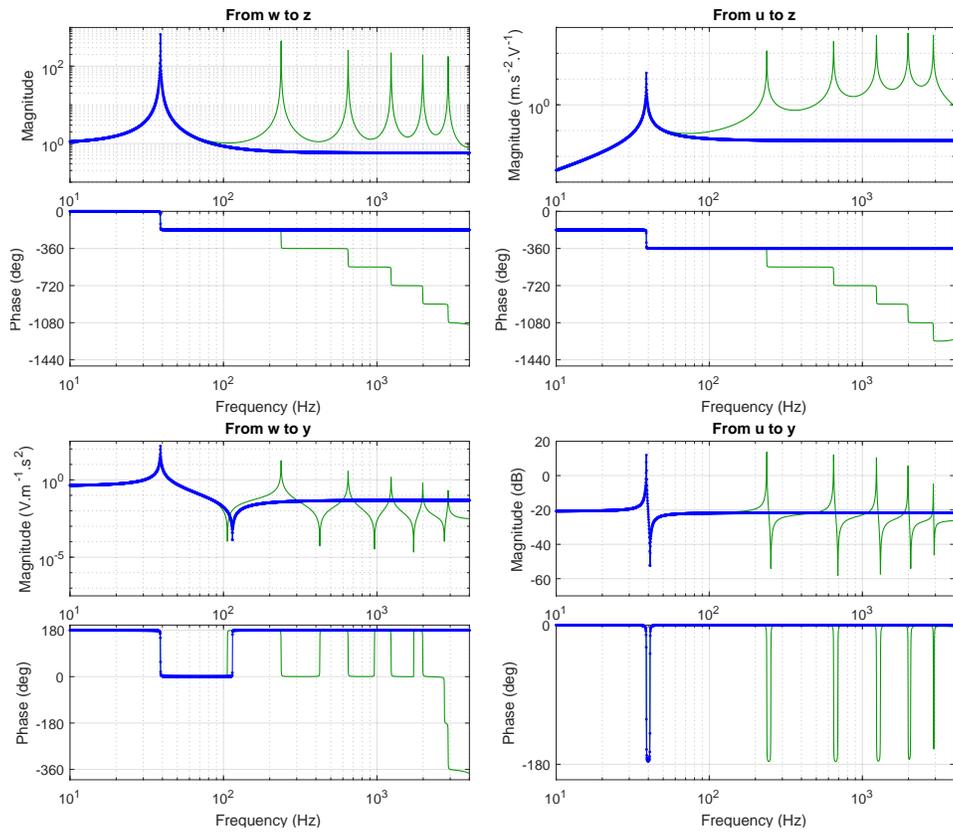
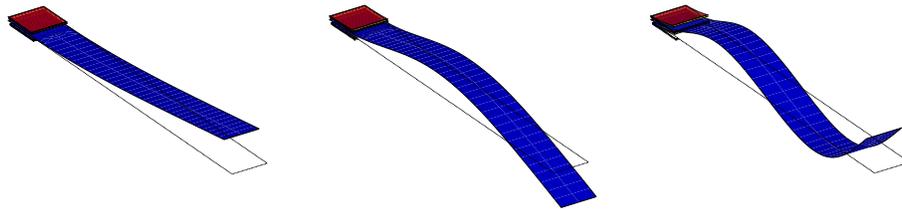


Figure 3: Frequency responses of the full-order model (green —) & design model (blue —).



(a) 1<sup>st</sup> bending mode at 38.8 Hz. (b) 2<sup>nd</sup> bending mode at 239 Hz. (c) 3<sup>rd</sup> bending mode at 649 Hz.

Figure 4: First three controllable and observable vibration modes.

filter combined with a QPB controller, with a low-frequency unitary gain but with a cutoff frequency left free for the design procedure. More precisely, the dynamical QPB controller

in (10) is sought with

$$\begin{aligned} D_0(s) &:= d_0(1 + \alpha s), \\ D_{\tau_0}(s) &:= d_{\tau_0}, \\ N_0(s) &:= n_0, \\ N_{\tau_0}(s) &:= n_{\tau_0}. \end{aligned}$$

It has  $N_P = 5$  independent parameters.

Using the notations of Sec. 3 and the design model's data, the resulting polynomials of the system's characteristic function  $\Delta(s)$  in (2) are:

$$\begin{aligned} P_0(s) &= \alpha d_0 s^3 + ((1 + \alpha a_1)d_0 - n_0 b_{uy_2}) s^2 + ((a_1 + \alpha a_0)d_0 - n_0 b_{uy_1}) s + d_0 a_0 - n_0 b_{uy_0}, \\ P_{\tau}(s) &= (d_{\tau_0} - n_{\tau_0} b_{uy_2}) s^2 + (d_{\tau_0} a_1 - n_{\tau_0} b_{uy_1}) s + (d_{\tau_0} a_0 - n_{\tau_0} b_{uy_0}). \end{aligned}$$

Notice that  $\Delta(s)$  is here of retarded type with  $m = 2$  and  $n = 3$ , thanks to the presence of the first-order filter. Let  $s_0 \in \mathbb{R}_-$  be the multiple root to be assigned. The total amount of independent parameters to be tuned is  $N_P = 5$ . Here, we have imposed  $\alpha = 1/d_0$ . As mentioned in Remark 2.1, the targeted multiplicity  $M(s_0)$  has been taken lower than the quasipolynomial's degree,  $5 = N_P \geq M(s_0) = 5 < \deg(\Delta(s)) = 6$ , in order to offer more possibility to assign  $s_0$  while giving enough equations to deal with the number of unknown parameters.

Hence, the controller's parameters and the multiple root are obtained by solving the set of equations given by (15) for  $k = 1$  to  $M(s_0) = 5$ .

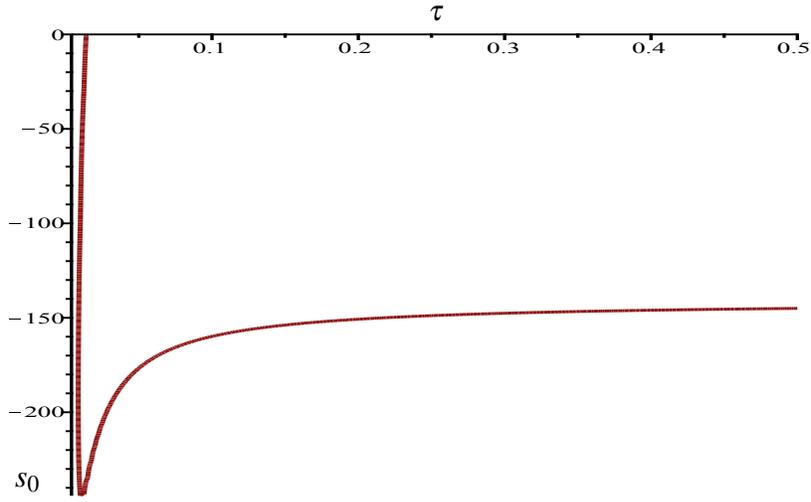
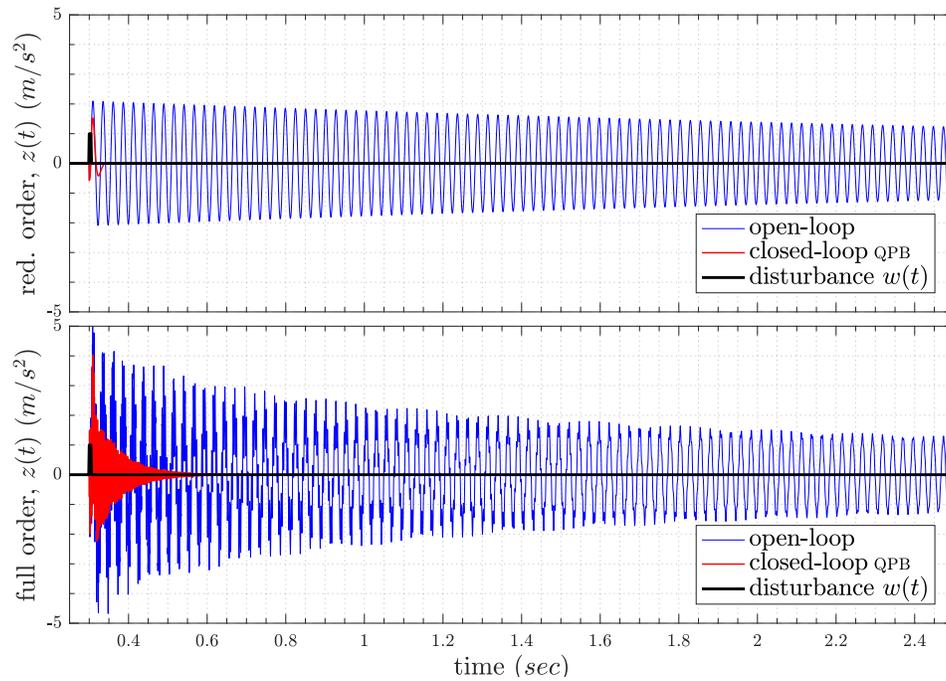
**4.3. Numerical and simulation results.** Given the numerical data of the design model indicated in [8], the numerical values of the dynamical QPB controller are given in Table 2, for  $s_0 = -220$ . The choice of this value is driven by twofold. It is selected in the

Table 2: Numerical results in the case  $s_0 = -220$ .

$n_0 \approx 10182.71$	$n_{\tau_0} \approx -7611.07$	$\tau \approx 8.9366 \cdot 10^{-3}$
$d_0 \approx 895.519$	$d_{\tau_0} \approx -637.158$	$\alpha \approx 1.1167 \cdot 10^{-3}$

admissible  $s_0(\tau)$ -curve in Fig. 5, derived from the previous design procedure. Among the admissible values, the one with a modulus close to the open-loop system's pole is preferred in order to reduce the control's effort in closed-loop. The roll-off filter's cutoff frequency is roughly equal to 142.5 Hz, clearly located at the right frequency region as usually set by the specialists of flexible structures' control.

Two simulations have been performed to check the closed-loop performances, each one with the design model first and then the full-order model. The one in Fig. 6 is the time response of the free-end's acceleration to the shock-like disturbance. In addition to the closed-loop stability for both models, one can notice the spectacular enhancement of the closed-loop settling time w.r.t. the open-loop. Fig. 7 shows a comparison of the frequency responses for the accelerometric transfer function,  $ie$  from input  $w$  to output  $w + z$ , in both cases: open-loop vs closed-loop. The damping of the first mode's peak of resonance has been successfully achieved, with an attenuation of more than 50 (dB). Moreover, this level of damping is maintained with the full-order model. It is worth mentioning that the neglected modes in the design model are remained stable and have also been damped.

Figure 5: Admissible  $(s_0, \tau)$  pair.Figure 6: Open (blue) vs closed-loop (red) time responses of the controlled output  $z$ , for the design model (top) and the full order model (bottom).

To conclude, the problem of disturbance rejection on the controlled output  $z(t)$  has been successfully addressed. The achieved closed-loop performances are very close to those in [30], obtained for the same system with an optimal  $\mathcal{H}_\infty$  controller of finite dimension,

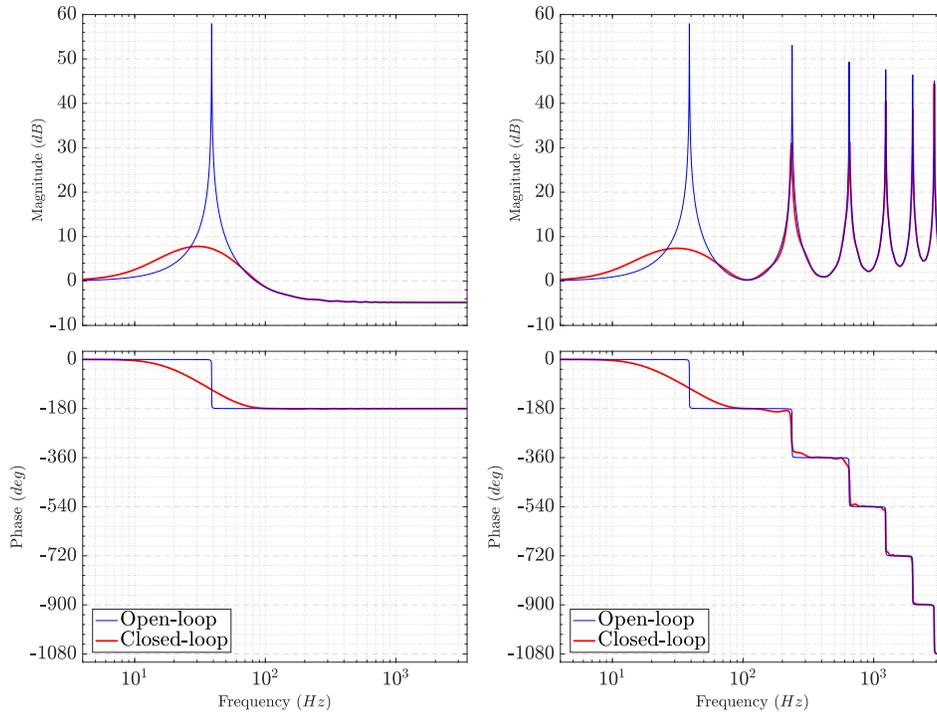


Figure 7: Open (blue) vs closed-loop (red) accelerometric frequency responses of the  $w$ -to- $(w+z)$  transfer function, for the design model (left) and the full order model (right).

designed with regional pole placement constraints and reduced to order 6. However, the structure of these controllers is very different.

## 5. CONCLUSION AND FUTURE WORKS

This work has shown the very promising features of an output-feedback controller based on a basic but tricky combination of delayed and filtered terms carrying on the input and output signals: the QPB controller with dynamical parameters where the time delay is used as a design parameter among the others. The appealing MID property has been cleverly adapted to offer a controller design procedure with enough degrees of freedom for the achievement of a given multiple and dominant root assignment in closed-loop. All of this has been illustrated by a realistic but challenging application of active vibration damping, though done in simulation. The next step will concern the experimental implementation of the QPB controller for some real-time tests. The challenge will consist of implementing the filtered delay actions in a reliable manner on Digital Signal Prototyping electronic board.

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