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# Stabilization of conservative mechanical systems in the presence of delay

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**Abstract:** In this study, we investigate the stabilization of conservative mechanical systems by delayed state feedback. The special structure of the open-loop characteristic polynomial allows us to analyze the location of the roots of a family of two-variable polynomials that is closely related to the characteristic quasipolynomial of the closed-loop system. In this way, we can determine the stabilizable intervals of the delay under some restrictions on the parameters in open-loop characteristic polynomial. The results are demonstrated through specific mechanical examples.

*Keywords:* conservative mechanical systems, time delay, dominant roots, spectral properties, state feedback

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## 1. INTRODUCTION

Mechanical systems are distinguished by their inherent structural characteristics, which arise from the fundamental physical laws governing their dynamic behavior, such as Newton's laws of motion, principles of energy conservation, and material properties (see, for instance, Seyranian and Mailybaev, 2003). The intrinsic structure of mechanical systems not only defines their operational behavior but also serves as a foundation for advanced analysis and mathematical modeling. Understanding these characteristics enables engineers and researchers to develop accurate predictive models that capture the system's dynamics under various operating conditions including linear and nonlinear regimes (Clark, 1996; Stépán, 1989).

In recent years, a novel paradigm of pole placement has emerged for infinite-dimensional systems, addressing the complexities posed by their infinite spectrum and intricate dynamics. This paradigm is fundamentally built upon the *multiplicity-induced dominance* (MID) of characteristic roots (see, for instance, Mazanti et al., 2021; Boussaada et al., 2024). While this property has proven to be a powerful tool in control design, its applicability often depends on specific system configurations and parameter constraints.

## 2. STABILIZATION OF CONSERVATIVE MECHANICAL SYSTEMS

Consider the linear delay differential equation of the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{C}\mathbf{K}_p\mathbf{q}(t - \tau) + \mathbf{C}\mathbf{K}_d\dot{\mathbf{q}}(t - \tau) \quad (1)$$

under appropriate initial conditions, where  $\mathbf{q}(t) \in \mathbb{R}^{m \times 1}$  is the unknown function,  $t \in \mathbb{R}$ ,  $\mathbf{M} \in \mathbb{R}^{m \times m}$  is a symmetric and positive definite mass matrix,  $\mathbf{S} \in \mathbb{R}^{m \times m}$  is a symmetric stiffness matrix,  $\mathbf{C} \in \mathbb{R}^{m \times 1}$ , the matrices  $\mathbf{K}_p$ ,  $\mathbf{K}_d \in \mathbb{R}^{1 \times m}$  are related to proportional-derivative feedback, and  $\tau > 0$  is the delay.

If  $\mathbf{K}_p = \mathbf{0}$  and  $\mathbf{K}_d = \mathbf{0}$ , then (1) takes the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{0}. \quad (2)$$

If there are only potential forces in a holonomic scleronomous mechanical system of finite degrees of freedom, and the potential energy does not explicitly depend on time (i.e., if the mechanical system is *conservative*), then we obtain an equation of motion of the form (2) in the neighborhood of a position of equilibrium (Gantmacher, 1975). Note that in such mechanical systems, there are no dissipative forces, gyroscopic forces, or any other non-conservative forces.

The trivial solution of (1) is exponentially stable if and only if all the roots of its characteristic function have negative real parts (Michiels and Niculescu, 2007). The characteristic function of (1) can be given as

$$D(s) = P(s) + Q(s)e^{-s\tau}, \quad (3)$$

where

$$\begin{aligned} P(s) &= a_{2m}s^{2m} + a_{2m-2}s^{2m-2} + \dots + a_0, \\ Q(s) &= b_{2m-1}s^{2m-1} + b_{2m-2}s^{2m-2} + \dots + b_0, \end{aligned} \quad (4)$$

and  $a_{2m} > 0$ . More precisely, the polynomial  $P(s)$  takes the form

$$P(s) = \det(s^2\mathbf{M} + \mathbf{S}). \quad (5)$$

The polynomial  $P(s)$  is a real-rooted polynomial in  $s^2$ , that is,  $P(s) = \widehat{P}(s^2)$  such that the polynomial

$$\widehat{P}(x) = \widehat{a}_m x^m + \widehat{a}_{m-1} x^{m-1} + \dots + \widehat{a}_0 \quad (6)$$

has only real roots,  $\widehat{a}_m > 0$ , and  $m \in \mathbb{Z}^+$ . The polynomial  $P(s)$  has only real roots if and only if every root of  $\widehat{P}(x)$  is nonnegative, that is, if and only if  $\mathbf{S}$  is negative semidefinite. In the latter case, the results of Balogh et al. (2022) apply. In any other case, further investigation is needed.

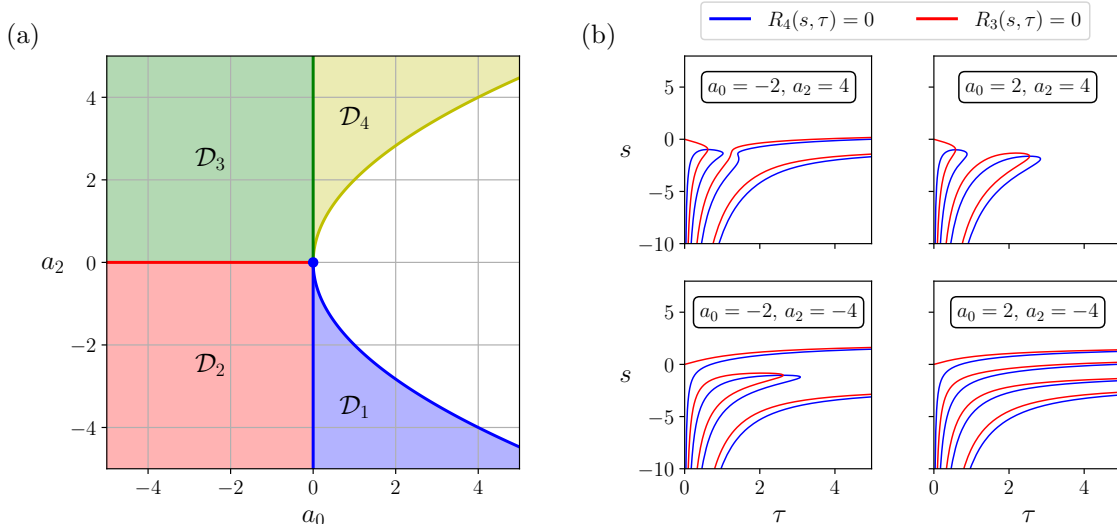


Fig. 1. Partition of the domain  $\mathcal{D}$  of parameters  $a_0$  and  $a_2$  (a). Branches of the real algebraic curves  $R_4(s, \tau) = 0$  and  $R_3(s, \tau) = 0$  for different values of  $a_0$  and  $a_2$  (b).

### 2.1 Problem formulation

A polynomial or quasipolynomial is called  $\gamma$ -stable if the real parts of all the roots of that polynomial or quasipolynomial are less than  $\gamma$ . Assume that the coefficients  $a_i$  in  $P(s)$  are known and fixed. Furthermore, assume that the coefficients  $b_i$  in  $Q(s)$  are independently adjustable control parameters. Then, the quasipolynomial  $D(s)$  is said to be  $\gamma$ -stabilizable for a given  $\tau$  if there exists a set of control parameters  $b_i(a_0, a_2, \dots, a_{2m}, \tau)$ ,  $i = 0, 1, \dots, 2m - 1$  for which  $D(s)$  is  $\gamma$ -stable.

In this study, we assess the  $\gamma$ -stabilizability of (3) by investigating the location of the roots of the family of two-variable polynomials

$$R_k(s, \tau) = \sum_{i=0}^k \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \quad k \in \mathbb{Z}_0^+. \quad (7)$$

### 2.2 Example

To illustrate the structure of polynomials (5) and (7), first, we consider the case where  $m = 2$  and  $a_{2m} = 1$ . Then, the quasipolynomial (3) takes the form

$$D(s) = s^4 + a_2 s^2 + a_0 + e^{-s\tau} (b_3 s^3 + b_2 s^2 + b_1 s + b_0), \quad (8)$$

where  $\tau > 0$  and  $a_2^2 - 4a_0 \geq 0$ .

We can define a partition of the domain

$$\mathcal{D} = \{(a_0, a_2) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0\} \quad (9)$$

of parameters  $a_0$  and  $a_2$  as shown in Figure 1a. Figure 1b shows the branches of the real algebraic curves  $R_4(s, \tau) = 0$  and  $R_3(s, \tau) = 0$  for different values of  $a_0$  and  $a_2$ . As can be seen, the polynomials  $R_4(s, \tau)$  and  $R_3(s, \tau)$  have a strictly dominant real root for  $s$  for every  $\tau > 0$  in the subdomains  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . This property proves to be essential for deriving the results of this study.

## 3. RESULTS AND DISCUSSION

As the main result of this study, we derive the  $\gamma$ -stabilizable intervals of  $\tau$  for the general class of quasipolynomials (3) with (4)–(5) under some restrictions on the

coefficients of the polynomials (6). At the same time, the results give a constructive method for assigning a dominant root of the quasipolynomial (3) by appropriately choosing the control parameters  $b_i$  in (4).

The results can be applicable in cases where the polynomial  $P(s)$  has non-real roots. In this sense, we extend the results of Balogh et al. (2022). This improvement is demonstrated through specific mechanical examples. On the other hand, the limitations of the results are shown in the case of the simple harmonic oscillator, where the polynomial  $P(s)$  has only purely imaginary roots.

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