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Towards a Partial Pole Placement for Systems of Differential Equations with Commensurate Delays: The Planar System Case*

Islam Boussaada^{1,2,3} and Fazia Bedouhene⁴ and Silviu-Iulian Niculescu¹

Abstract—Recent studies have introduced a stabilization framework entitled *Partial Pole Placement (PPP)* for linear time-invariant single-input/single-output (LTI SISO) systems with input delay. By exploiting the delay as a *control parameter*, this approach enables the precise prescription of the closed-loop solution’s exponential decay rate by directly assigning the spectral abscissa of the associated quasipolynomial. The foundation of PPP lies in a spectral property called *Multiplicity-induced-dominancy (MID)*. For quasipolynomials with a single delay, the generic MID (GMID) property serves as both a cornerstone and the most striking manifestation of the MID principle, asserting that a spectral value with maximal multiplicity is *necessarily dominant*. However, a key limitation of the GMID is its restricted parametric flexibility, which constrains its applicability to control design. In this paper, we show how to harness the MID more effectively by highlighting the advantages of intermediate multiplicities for control applications in multiple-input/multiple-output (MIMO) systems with input delay. To this end, the validity of the intermediate (over-order) MID (IMID) property is investigated, then exploited in the design of a state-feedback achieving a prescribed exponential stabilization of the solution of planar dynamical systems with input delay.

Index Terms—Multiple-input/multiple-output (MIMO) systems; delay; spectral methods; pole placement; commensurate delays; multiplicity-induced-dominancy (MID).

I. INTRODUCTION AND MOTIVATIONS

Delay-differential equations (DDEs) arise naturally in the modeling of diverse dynamical phenomena, including transport, propagation, communication, and hereditary processes. Due to their dependence on past states, DDEs are inherently infinite-dimensional, and their analysis and control require tools beyond those used in standard finite-dimensional systems. For an introduction to the basic theory of DDEs, we refer to [1]–[3], while more advanced treatments and applications are detailed in [4]–[7].

In the linear case, DDEs are characterized by quasipolynomials whose infinitely many (characteristic) roots depend non-trivially on system parameters, especially delay. A key quantity in stability analysis is the spectral abscissa, defined as the real

part of the rightmost root. Understanding its dependence on parameters is essential in control design [6].

It has long been known, since Hazen’s early work in the 1930s [8] that delays can induce instability or degrade performance. However, under suitable conditions, delay can act as a *design parameter* to enhance stabilization, particularly in low-order controller configurations [6], [7].

The pole placement problem for delay systems remains challenging due to the transcendental nature of the characteristic function. Full spectral assignment is infeasible; instead, control design often focuses on assigning a subset of dominant spectral values. A key structural insight is the connection between high-multiplicity spectral values and optimal decay rates, identified by Pinney [9] and more recently formalized in [10], which showed that in the planar single-delay case, spectral values with maximal multiplicity coincide with global minimizers of the spectral abscissa.

This observation led to the notion of *multiplicity-induced dominancy (MID)* introduced in [11], which asserts that, in generic quasipolynomials, a characteristic root of maximal admissible multiplicity determines the spectral abscissa [12], [13], a property called GMID. A more practical variant is the *Intermediate MID* property (IMID for short), which involves lower multiplicities but still preserves the dominance structure necessary for the prescribed stabilization [14].

Building on these ideas, the Partial Pole Placement (PPP) approach has been developed to assign dominant roots in systems with input delays. In the single-input case, it has been fully characterized for arbitrary-order systems under both GMID [13] and IMID [14] settings. The key benefit of reducing multiplicity, while preserving the dominant root, is increased robustness and parametric flexibility. Lower multiplicities enlarge the set of admissible parameters and reduce sensitivity to perturbations, as multiple eigenvalue splitting under parameter changes follows Puiseux expansions [15].

Moreover, the MID-based PPP approach uniquely allows analytical certification of assigned spectral abscissa and in turn certifies the prescribed decay rate of the closed-loop solution. For instance, this was illustrated in [16], through the systematic tuning of the PID gains, achieving prescribed stabilization of unstable first-order systems while balancing robustness, delay margin, and tracking performance. In summary, these developments position MID-based PPP as a structured and effective methodology for stabilizing delay systems. This work aims to extend this framework by formalizing PPP for systems with *commensurate delays* under *IMID conditions*, offering new theoretical insights and practical design tools for control.

The contribution of the paper is threefold: first, we address

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the MID property for linear planar systems including two commensurate delays recovering the GMID as well as the codimension one IMID. To the best of the authors' knowledge such a result represents a novelty in the literature. Second, the derived IMID property is explicitly used in constructing an appropriate pole placement controller showing the advantages as well as the limitations that we may have in terms of delays and spectrum location. Such discussions open new perspectives for a better understanding of the way to use *delay* as a *control parameter*. Finally, the derived results are applied to the control of the classical inverted pendulum using the proposed pole placement methodology and revisiting some of the existing results [17] in the open literature in a different framework.

The remaining of the paper is organized as follows: Section II recalls some important facts and describes the problem addressed in the paper. Next, the main results are presented and discussed in Section III and illustrated in the case of the inverted pendulum in Section IV. The paper ends with some concluding remarks. The notations are standard and explained when firstly used.

II. PREREQUISITES AND PROBLEM STATEMENT

Consider the state-feedback stabilization problem of the planar linear-time invariant (LTI) system with input delay:

$$\dot{\xi}(t) = \mathbf{A}_0 \xi(t) + \mathbf{B} u(t - \sigma) \quad (1)$$

where $\xi(t) = (\xi_1(t), \xi_2(t))^T \in \mathbb{R}^2$ denotes the state vector, $u(t) = (u_1(t), u_2(t))^T \in \mathbb{R}^2$ the input, with matrices $\mathbf{A}_0, \mathbf{B} \in \mathcal{M}_2(\mathbb{R})$ and $\sigma \in \mathbb{R}_+$ the (input) delay. As in the standard pole placement method, consider that the pair $(\mathbf{A}_0, \mathbf{B})$ is *controllable*. In the free-delay case, the controllability property ensures that the roots of the characteristic function of the corresponding finite-dimensional system can be arbitrarily assigned by static-state feedback guaranteeing an appropriate decay rate of the solutions of the system in closed-loop by choosing the location of the *spectral abscissa*¹. In other words, there exists a real matrix $\mathbf{K} \in \mathcal{M}_2(\mathbb{R})$ such that the control law:

$$u(t) = \mathbf{K} \xi(t) \in \mathbb{R}^2 \quad (2)$$

guarantees the requested location of the spectrum of the closed-loop system. Unfortunately, such a property is not valid if there exists an input delay. In fact, even for a small delay, the characteristic function of the closed-loop system controlled by a static-state feedback becomes a *quasipolynomial* with an infinite number of (characteristic) roots, whose location cannot in general be "fully" assigned by using a finite number of (control) gains. As indicated in the Introduction, it should be emphasized that the delay, seen as a *control parameter*, allows more degree of freedom to assign the spectral abscissa of the closed-loop system.

Under an appropriate initial condition defined in the Banach space $C([- \sigma, 0], \mathbb{R}^2)$, the existence and uniqueness of the solution of the system (3) is guaranteed (see, for instance,

[3]). Without any loss of generality, the closed-loop system (1)-(2) can be reduced to its canonical form defined as:

$$\dot{\xi}(t) = \mathbf{A}_0 \xi(t) + \mathbf{A}_\sigma \xi(t - \sigma), \quad (3)$$

with matrices \mathbf{A}_0 in companion form and \mathbf{A}_σ such that:

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}, \quad \mathbf{A}_\sigma = \mathbf{B}\mathbf{K} = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix}. \quad (4)$$

Observe that, despite the presence of a single delay in (3), the corresponding characteristic function includes *two commensurate delays*. In fact, the corresponding characteristic quasipolynomial function $\Delta : \mathbb{C} \rightarrow \mathbb{C}$ writes as: $\Delta(s) = P_0(s) + P_1(s)e^{-\sigma s} + P_2(s)e^{-2\sigma s}$, where

$$\begin{cases} P_0(s) = s^2 - bs - a, \\ P_1(s) = (-k_{1,1} - k_{2,2})s - ak_{1,2} + bk_{1,1} - k_{2,1}, \\ P_2(s) = k_{1,1}k_{2,2} - k_{1,2}k_{2,1}. \end{cases} \quad (5)$$

For the sake of brevity, and in order to simplify the presentation in the sequel, denote $a_{i,j}$ for $0 \leq i, j \leq 2$ the coefficient of s^j of P_i in (5). It is worth noting that parameters in (5) can be identified from $a_{i,j}$ as follows: $a = -a_{0,0}, b = -a_{0,1}, k_{2,1} = a_{0,0}k_{1,2} - a_{0,1}k_{1,1} - a_{1,0}, k_{2,2} = -k_{1,1} - a_{1,1}$ where $k_{1,1}$ is defined as one of the two real roots of $k_{1,1}^2 + (-a_{0,1}k_{1,2} + a_{1,1})k_{1,1} + a_{0,0}k_{1,2}^2 - a_{1,0}k_{1,2} + a_{2,0} = 0$, which exist if the free parameter $k_{1,2}$ is chosen such that the polynomial $(a_{0,1}^2 - 4a_{0,0})k_{1,2}^2 + (-2a_{1,1}a_{0,1} + 4a_{1,0})k_{1,2} + a_{1,1}^2 - 4a_{2,0}$ is positive.

Notice that, in the particular configuration where $\det(\mathbf{A}_\sigma)$ vanishes (that is $a_{2,0} = 0$), then $\Delta(s) = P_0(s) + P_1(s)e^{-\sigma s}$, in that case, the over-order MID property has been fully characterized in [11], [13], [14]. However, the only few results on the validity of the MID property for quasipolynomials with two delays were recently established in [18]–[20] where only the *GMID* characterization was proposed. More precisely, the following result from [19] asserts the validity of the *GMID* for the quasipolynomial Δ , given by

$$\Delta(s) = s^2 + a_{0,1}s + a_{0,0} + (a_{1,1}s + a_{1,0})e^{-\sigma s} + a_{2,0}e^{-2\sigma s} \quad (6)$$

with $a_{1,1} \neq 0$ and $a_{2,0} \neq 0$.

Theorem 1 ([19]). *Consider the system (3)-(4) and its corresponding quasipolynomial (6). Then*

1) s_0 is a real root of Δ with maximal multiplicity 5 if, and only if, the following conditions are satisfied:

$$\begin{cases} s_0 &= \frac{1}{2\sigma}(-a_{0,1}\sigma - 5), \quad a_{0,0} = \frac{1}{4\sigma^2}(a_{0,1}^2\sigma^2 + 9), \\ a_{1,0} &= -\frac{2}{\sigma^2}(a_{0,1}\sigma + 9)e^{-\frac{1}{2}(5+a_{0,1}\sigma)}, \\ a_{1,1} &= -\frac{4}{\sigma}e^{-\frac{1}{2}(5+a_{0,1}\sigma)}, \quad a_{2,0} = \frac{-1}{2\sigma^2}e^{-\frac{1}{2}(5+a_{0,1}\sigma)}. \end{cases} \quad (7)$$

2) The root s_0 is a strictly dominant root of Δ .

3) Moreover, if $s_0 < 0$, then the trivial solution of (3)-(4) is exponentially stable with the decay rate $-s_0$.

Remark 1. *Clearly, the structural relationship between (5) and (6), as well as the requirement in (7) that $a_{0,0}$ be positive, shows that Theorem 1 is ineffective in controlling the system (3)–(4) when $a > 0$. To enhance the efficiency of the partial pole placement, additional parametric freedom is needed to avoid constraints on the open-loop parameters or*

¹For a characteristic function, the *spectral abscissa* denotes the real part of the rightmost root of the spectrum; see, e.g., [6] and the references therein

the delay. While greater parametric flexibility broadens the range of feasible solutions in control design, reducing the dominant spectral value multiplicity enhances the controller robustness in the presence of parametric uncertainties; see [15] for further details on multiple spectral values sensitivity with respect to parametric uncertainties.

The objective of this work is to propose a solution that simultaneously addresses these issues by reducing multiplicity while preserving the dominance property, which in turn ensures the desired prescribed stabilization.

The proof of Theorem 1 relies on the argument principle applied to the normalized quasipolynomial,

$$\tilde{\Delta}_5(z) = (-8z - 32)e^{-\frac{z}{2}} + z^2 - 10z - 2e^{-z} + 34. \quad (8)$$

This maps the root s_0 of multiplicity 5 of Δ to $z = 0$ for $\tilde{\Delta}_5$. The analysis relies on refinements of the Stépán-Hassard argument variation, which will also be useful later on.

Theorem 2 ([4], [21]). *Consider the linear delay-differential system $\dot{x}(t) = \sum_{j=1}^J A_j x(t - \tau_j)$, where for each $j = 1, \dots, J$, $\tau_j \geq 0$ and $A_j \in \mathbb{R}^{n \times n}$ is a real matrix. Let $D(s) = \det(sI - \sum_{j=1}^J e^{-s\tau_j} A_j)$. Let*

$$R(\omega) = \Re(i^{-n} D(i\omega)), \quad I(\omega) = \Im(i^{-n} D(i\omega)). \quad (9)$$

Let $\omega_1, \dots, \omega_r$ be the positive zeros of $R(\omega)$, counted with multiplicity and ordered so that $0 < \omega_r \leq \dots \leq \omega_1$. For each $j = 1, \dots, r$ such that $D(i\omega_j) = 0$, assume that the multiplicity of $i\omega_j$ as a zero of $D(s)$ is the same as the multiplicity of ω_j as a zero of $R(\omega)$. Then, the number $\nu_+(D)$ of roots of $D(s) = 0$, which lie in $\Re(s) > 0$, counted with multiplicity, is given by:

$$\nu_+(D) = \frac{n - K}{2} + \frac{(-1)^r}{2} \operatorname{sgn}(I^{(m)}(0)) + \sum_{j=1}^r (-1)^{j-1} \operatorname{sgn}(I(\omega_j)), \quad (10)$$

where n is the system order, K the total multiplicity of roots on $\Re(s) = 0$, m the multiplicity of $s = 0$, and r the number of positive roots of $R(\omega)$.

Furthermore, $\nu_+(D)$ is odd (even) if $D^{(m)}(0) < 0$ ($D^{(m)}(0) > 0$). If $R(y)$ has no positive zeros, set $r = 0$ and omit the summation term.

III. MAIN RESULT

In this section, we derive the conditions on the coefficients of the quasipolynomial Δ that ensure the existence of a real multiple root s_{\pm} with multiplicity 4, thereby enhancing the effectiveness of the resulting assignment.

Before stating the main result, denote

$$\kappa = W\left(\frac{1}{4}\sigma a_{1,1} e^{\frac{1}{2}(\sigma a_{0,1} + 3)}\right), \quad (11)$$

where κ admits two real branches $\kappa = \kappa_{\pm}$ if, and only if,

$$\sigma a_{1,1} \geq -4e^{-\frac{1}{2}(\sigma a_{0,1} + 3)}, \quad (12)$$

where $\kappa_+ = W_0$ is the principal real branch and $\kappa_- = W_{-1}$ is the lower real branch of the Lambert W function, see for instance [22]. It is natural to assume that this condition on $a_{1,1}$

and $a_{0,1}$ holds in the following. Notice also that the limiting case corresponds to the configuration of Theorem 1.

The next theorems provide *necessary and sufficient conditions* guaranteeing the negativity of a spectral value with multiplicity 4, along with a sufficient condition guaranteeing its dominance. Moreover, it explicitly determines the MID-based controller's gains and delay by leveraging a lower multiplicity.

Theorem 3. *For an arbitrary positive delay σ , the quasipolynomial Δ admits a real spectral value at $s = s_{\pm}$ with algebraic multiplicity 4 if, and only if,*

$$\begin{cases} s_+ = \frac{1}{2\sigma}(-\sigma a_{0,1} - 3 + 2\kappa_+), & \text{if } a_{1,1} \neq 0 \\ s_- = \frac{1}{2\sigma}(-\sigma a_{0,1} - 3 + 2\kappa_-), & \text{if } a_{1,1} < 0. \end{cases} \quad (13)$$

and the system parameters satisfy:

$$\begin{cases} a_{0,0} = \frac{1}{4\sigma^2}(\sigma^2 a_{0,1}^2 - 4\kappa_{\pm}^2 - 8\kappa_{\pm} + 5), \\ a_{1,0} = -\frac{a_{1,1}}{2\sigma\kappa_{\pm}}(-\kappa_{\pm}\sigma a_{0,1} + 2\kappa_{\pm}^2 - 5\kappa_{\pm} + 2), \\ a_{2,0} = \frac{1}{32\kappa_{\pm}^2}(2\kappa_{\pm} + 1)a_{1,1}^2. \end{cases} \quad (\star_{\pm})$$

with κ being $\kappa = \kappa_{\pm}$, depending on the sign of $a_{1,1}$. Moreover, for any delay σ , the following inequality is satisfied: $s_- \leq s_0 \leq s_+$, where s_0 is the quintuple root established in (7) and equality occurs iff $\kappa = -1$. Furthermore, the value s_{\pm} is negative if, and only if, the rescaled parameters $\tilde{a}_{0,1} = \sigma a_{0,1}$ and $\tilde{a}_{1,1} = \sigma a_{1,1}$ satisfy one of the conditions: (C1) if $a_{1,1} > 0$, or (C2) if $a_{1,1} < 0$ and $\kappa = \kappa_-$, or (C3) if $a_{1,1} < 0$ and $\kappa = \kappa_+$, where:

$$(C1) : 0 < \tilde{a}_{1,1} < 2(\tilde{a}_{0,1} + 3),$$

$$(C2) : -4e^{-\frac{1}{2}(\tilde{a}_{0,1} + 5)} < \tilde{a}_{1,1} < \min\{2(\tilde{a}_{0,1} + 3), 0\},$$

$$(C3) : 0 > \tilde{a}_{1,1} > \max\{2(\tilde{a}_{0,1} + 3), -4e^{-\frac{1}{2}(\tilde{a}_{0,1} + 5)}\}.$$

Proof: **Interpolation conditions:** Thanks to the linear dependency of the quasipolynomial Δ in the parameters $a_{i,j}$, solutions of the system of transcendental equations $\frac{d^k \Delta}{ds^k}(s) = 0$, $k = 0, \dots, 3$ reduce to solutions of an algebraic system allowing to (13) and (\star_{\pm}) relating $a_{0,0}$, $a_{1,0}$, $a_{2,0}$ and s_{\pm} , as functions of $a_{0,1}$, $a_{1,1}$, σ , and κ . Inequality $s_- \leq s_0 \leq s_+$ follows directly from the fact that $\kappa_+ \geq -1 \geq \kappa_-$ and from the observation that when $\kappa = -1$, we recover s_0 , which is the root of the quasipolynomial with maximal multiplicity 5, as stated in Theorem 1, see [19]. The value $a_{1,1} = 0$ is excluded in order to ensure that Δ has degree 5.

Negativity conditions: If $\tilde{a}_{1,1} < 0$, and $\kappa = \kappa_- < -1$, from the expression of s_- , we have $s_- < 0$ iff $-\tilde{a}_{0,1} - 3 + 2\kappa_- < 0$. Using the Lambert W function, we obtain:

$$s_- < 0 \iff W_{-1}\left(\frac{1}{4}\tilde{a}_{1,1} e^{\frac{1}{2}(\tilde{a}_{0,1} + 3)}\right) < \frac{1}{2}(\tilde{a}_{0,1} + 3).$$

Exploiting the decreasing and the upper-boundedness of W_{-1} , that is $W_{-1} < -1$ leads to the following inequality:

$$\frac{\tilde{a}_{1,1}}{4} e^{\frac{1}{2}(\tilde{a}_{0,1} + 3)} > \max\left\{\frac{1}{2}(\tilde{a}_{0,1} + 3) e^{\frac{1}{2}(\tilde{a}_{0,1} + 3)}, -e^{-1}\right\}.$$

Finally, isolating $\tilde{a}_{1,1}$ gives the desired condition:

$$0 > \tilde{a}_{1,1} > \max\left\{2(\tilde{a}_{0,1} + 3), -4e^{-\frac{1}{2}(\tilde{a}_{0,1} + 5)}\right\}.$$

The other inequalities follow similarly by exploiting the bijectivity of the real branches of the Lambert W function. \square

Remark 2. Notice that (\star_+) provides a condition on $a_{0,1}$ and $a_{1,1}$, since κ depends on these 2 parameters. Moreover, we gain more flexibility in the choice of s_+ compared to s_0 , since $s_0 = \frac{1}{2\sigma}(-\sigma a_{0,1} - 5)$, see (7), while s_+ lies in the half-plane defined by $s_+ \geq \frac{1}{2}(-\tilde{a}_{0,1} - 5)$. From a control perspective, the first equality in (\star_+) allows first, if $(a_{0,1}^2 - 4a_{0,0})\sigma^2 + 9 > 0$, to solve κ and then thanks to (11) to obtain an explicit expression of $a_{1,1}$ as a function of the open-loop parameters $a_{0,0}$, $a_{0,1}$ and the delay σ . This fact will be further highlighted in Section IV.

Introducing the parameter $\mu = 2(\kappa_{\pm} + 1)$, we establish the validity of the IMID property of the root s_+ when μ varies within a certain bounded interval. More precisely, we have:

Theorem 4 (Dominance of s_+). If (\star_+) (respectively (\star_-)) holds, then there exists $\mu^* > 0$ such that $s = s_+$ represents the spectral abscissa associated with (6) for all $\mu \in [0, \mu^*]$ (while s_- cannot correspond to the spectral abscissa of (6)).

Proof: The proof follows the same steps as in [11] but adapted to two commensurate delays. To determine the value of μ guaranteeing the dominance of s_+ , we apply the Stépán-Hassard Theorem 2. Define the following parametrization

$$\mu = 2(\kappa_{\pm} + 1); \quad s = \frac{z}{2\sigma} + s_{\pm}. \quad (14)$$

Incorporating the coefficients from (\star_{\pm}) and the parameterization (14) into $\Delta(s)$ yields the normalized quasipolynomial:

$$\begin{aligned} \tilde{\Delta}_4(z) = & \left(2z + 2e^{-z} + e^{-\frac{1}{2}z}(4z + 8) - 10\right) \mu \\ & + \left(z^2 - 10z + 34 - e^{-\frac{1}{2}z}(8z + 32) - 2e^{-z}\right). \end{aligned} \quad (15)$$

This transformation shifts the root s_+ to zero. When $\mu = 0$, we recover the quasipolynomial $\tilde{\Delta}_5$ given by (8), which corresponds to a root of multiplicity 5 at zero.

Moreover, since $\tilde{\Delta}_4 = \tilde{\Delta}_5 + \mu\tilde{\Delta}_5$, where the prime symbol denotes the derivative with respect to z , we can express the real and imaginary parts of $\tilde{\Delta}_4$ in terms of those of $\tilde{\Delta}_5$. Let

$$\begin{cases} \Re^{\mu}(\omega) = \Re\left(i^{-2}\tilde{\Delta}_4(i\omega)\right), & \Im^{\mu}(\omega) = \Im\left(i^{-2}\tilde{\Delta}_4(i\omega)\right), \\ \mathcal{R}(\omega) = \Re\left(i^{-2}\tilde{\Delta}_5(i\omega)\right), & \mathcal{I}(\omega) = \Im\left(i^{-2}\tilde{\Delta}_5(i\omega)\right), \end{cases}$$

Then, we have:

$$R^{\mu}(\omega) = -\mathcal{I}'(\omega)\mu + \mathcal{R}(\omega), \quad I^{\mu}(\omega) = -\mathcal{R}'(\omega)\mu - \mathcal{I}(\omega).$$

The explicit expressions of $\mathcal{R}(\omega)$ and $\mathcal{I}(\omega)$, whose variation has been investigated in [19], are given by:

$$\begin{aligned} \mathcal{R}(\omega) &= \omega^2 + 8 \sin\left(\frac{\omega}{2}\right)\omega + 32 \cos\left(\frac{\omega}{2}\right) + 2 \cos(\omega) - 34, \\ \mathcal{I}(\omega) &= 8 \cos\left(\frac{\omega}{2}\right)\omega + 10\omega - 32 \sin\left(\frac{\omega}{2}\right) - 2 \sin(\omega). \end{aligned}$$

In particular, recall some important properties of $\mathcal{I}'(\cdot)$ and $\mathcal{R}'(\cdot)$ as emphasized in [19]. There exists a unique solution $\omega_{\mathcal{I}'=0}^1 \in (\pi, \frac{3\pi}{2})$, the root of $\mathcal{I}'(\omega) = 0$. Moreover, $\mathcal{I}'(4\pi k) = 0, \forall k \geq 1$, and there exists a unique $\omega_{\mathcal{I}'=0}^{2,k} \in \left(\frac{(8k+3)\pi}{2}, (4k+2)\pi\right)$ for each $k \geq 1$, such that $\mathcal{I}'(\omega_{\mathcal{I}'=0}^{2,k}) =$

0. These points define the vertical asymptotes of ϱ_R . Similarly, the roots of $\mathcal{R}'(\omega)$ are located in the intervals: $\omega_{\mathcal{R}'=0}^1 \in (2\pi, 3\pi)$, $\omega_{\mathcal{R}'=0}^{2,k} \in ((4k+1)\pi, (4k+2)\pi)$, $\omega_{\mathcal{R}'=0}^{3,k} \in ((4k+2)\pi, (4k+3)\pi)$. They define the vertical asymptotes of ϱ_I , with the objective to identify the points $\omega > 0$ for which

$$\varrho_R(\omega) = \frac{\mathcal{R}(\omega)}{\mathcal{I}'(\omega)} \quad \text{and} \quad \varrho_I(\omega) = -\frac{\mathcal{I}(\omega)}{\mathcal{R}'(\omega)}$$

both equal μ . These values determine the roots of R^{μ} . Regarding the sign of I^{μ} at a root ω^* of R^{μ} , we have:

$$I^{\mu}(\omega^*) > 0 \iff \begin{cases} \varrho_R(\omega^*) > \varrho_I(\omega^*) & \text{if } \mathcal{R}'(\omega^*) < 0, \\ \varrho_R(\omega^*) < \varrho_I(\omega^*) & \text{if } \mathcal{R}'(\omega^*) > 0. \end{cases} \quad (16)$$

$$\text{sgn}(I^{\mu}(\omega^*)) = -\text{sgn}(\mathcal{R}'(\omega^*)) \cdot \text{sgn}(\varrho_R(\omega^*) - \varrho_I(\omega^*)). \quad (17)$$

Taking into account the variations of \mathcal{R}' , \mathcal{I}' , \mathcal{R} , and \mathcal{I} as established in [19], we deduce that ϱ_R is extendable by continuity at 0, with $\varrho_R(0) = 0$. Moreover, on $(0, \omega_{\mathcal{I}'=0}^1)$, the function ϱ_R is decreasing from 0 to $-\infty$. On $(\omega_{\mathcal{I}'=0}^1, 4\pi)$, the function ϱ_R is decreasing from 0 to $-\infty$, passing through the unique point $\{(\omega_{\mathcal{R}=0}, 0)\}$ on the horizontal axis.

Furthermore, if $\mu_1, \mu_3, \dots, \mu_{2k+1}, \dots$ are such that $\mu_k = \varrho_R^+(\omega_{\varrho_R=0}^k)$, where ϱ_R^+ is the positive part of ϱ_R , $\omega_{\varrho_R=0}^k \in (4k\pi, \omega_{\mathcal{I}'=0}^{2,k})$ and $\varrho_R'(\omega_{\varrho_R=0}^k) = 0$, then $(\mu_{2k+1})_k$ is increasing (the increasing behavior stems from the properties of \mathcal{R} and \mathcal{I}). Moreover, ϱ_R is decreasing on $(4k\pi, \omega_{\varrho_R=0}^k)$ and increasing on $(\omega_{\varrho_R=0}^k, \omega_{\mathcal{I}'=0}^{2,k})$, for all $k \geq 1$.

In addition, ϱ_R and ϱ_I intersect at points $\omega_{\varrho_R=\varrho_I}^k \in ((4k+1)\pi, (4k+2)\pi)$, for $k \geq 1$, since

$$(\varrho_R - \varrho_I)((4k+1)\pi) \cdot (\varrho_R - \varrho_I)((4k+2)\pi) < 0, \quad \forall k \geq 1.$$

If $\mu_2, \mu_4, \dots, \mu_{2k}, \dots$ are such that $\varrho_R(\omega_{\varrho_R=\varrho_I}^k) = \mu_{2k}$, then $(\mu_{2k})_k$ is increasing. Moreover, thanks once again to the properties of \mathcal{R} and \mathcal{I} , we have $\mu_{2k+1} < \mu_{2k}$, for all $k \geq 1$. We deduce the following after defining the roots of $\mu = \varrho_R(\omega)$ as $\omega_{R^{\mu}}^k$ for $k \geq 1$:

1) For all $\mu \in (0, \mu_1)$, equation $\mu = \varrho_R(\omega)$ admits a unique solution $\omega_{R^{\mu}}^1 \in (\pi, \omega_{\mathcal{R}=0})$.

• If $\omega_{R^{\mu}}^1 \in (\pi, \omega_{\mathcal{I}'=0}^1) \subset (\pi, \omega_{\mathcal{R}=0})$, then $\mathcal{R}'(\omega_{R^{\mu}}^1) < 0$ and $\varrho_R(\omega_{R^{\mu}}^1) > \varrho_I(\omega_{R^{\mu}}^1)$. By virtue of (16), $\text{sgn}(I^{\mu}(\omega_1)) = +1$.

• If $\omega_{R^{\mu}}^1 \in [\omega_{\mathcal{I}'=0}^1, \omega_{\mathcal{R}=0})$, then $\mathcal{R}'(\omega_{R^{\mu}}^1) > 0$ and $\varrho_R(\omega_{R^{\mu}}^1) < \varrho_I(\omega_{R^{\mu}}^1)$. Thus, $\text{sgn}(I^{\mu}(\omega_{R^{\mu}}^1)) = +1$ for all $\mu \in (0, \mu_1)$.

It is worth noting that in the interval $(0, \mu_1)$, there is no intersection point between ϱ_R and ϱ_I . As a result, there is no $\omega > 0$ satisfying both $\mathcal{R}_{\mu}(\omega) = 0$ and $\mathcal{I}_{\mu}(\omega) = 0$.

Elementary calculations show that $I_{\mu}^{(4)}(0) = 0$. Consequently, the only ω satisfying $\tilde{\Delta}_4(i\omega) = 0$ with multiplicity 4 is $\omega = 0$. Thus, from (10), with $n = 2$, $K = 4$, $r = 1$, $m = 4$, we get $\nu_+(\tilde{\Delta}_4) = \frac{2-4}{2} + \frac{1}{2}(-1)^1 \cdot 0 + (-1)^0 \cdot 1 = 0$.

2) For $\mu = \mu_1$, the equation $\mu_1 = \varrho_R(\omega)$ admits three roots, including $\omega_{R^{\mu}}^1 \in (\pi, 2\pi)$ and the double root $\omega_{R^{\mu}}^2 = \omega_{\varrho_R=0}^1 = \arg \min_{\omega \geq 4\pi} \varrho_R^+(\omega)$, with $\text{sgn}(I^{\mu}(\omega_{R^{\mu}}^1)) = +1$ and $\text{sgn}(I^{\mu}(\omega_{R^{\mu}}^2)) = -1$. Thus, the Stépán-Hassard formula, with $K = 4$, $r = 3$, $m = 4$, gives in this case: $\nu_+(\tilde{\Delta}_4) = 0$.

3) For $\mu = \mu_2$, equation $\mu = \varrho_R(\omega)$ admits three distinct roots: $\omega_{R^{\mu}}^1 \in (\pi, 2\pi)$, $\omega_{R^{\mu}}^2 \in (4\pi, \omega_{\varrho_R=0}^1)$, $\omega_{R^{\mu}}^3 = \omega_{\varrho_R=0}^1 \in$

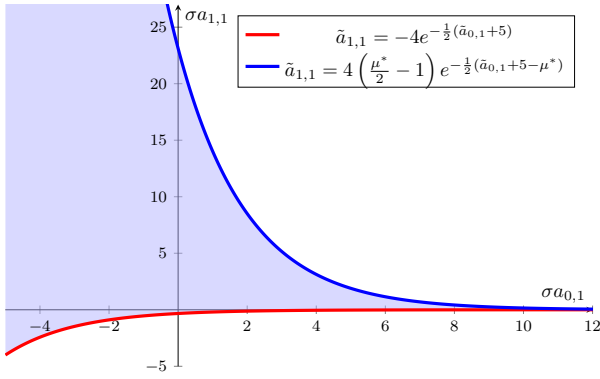


Fig. 1. Region $\mathcal{D}_{\text{IMID}}^+$ representing the parameter domain ensuring the dominance of the spectral value s_+ , depending on $\tilde{a}_{0,1}$ & $\tilde{a}_{1,1}$.

$(5\pi, 6\pi)$, with $\text{sgn}(I^\mu(\omega_{R^\mu}^1)) = +1$, $\text{sgn}(I^\mu(\omega_{R^\mu}^2)) = -1$, $\text{sgn}(I^\mu(\omega_{R^\mu}^3)) = 0$. Note that $\pm i\omega_{R^\mu}^3$ are two purely imaginary roots of $\tilde{\Delta}_4$. Thus, applying Theorem (2) with $K = 6$, $r = 3$ in (10), which gives $\nu_+(\tilde{\Delta}_4) = 0$.

4) For $\mu \in (\mu_1, \mu_2)$, the equation $\mu = \varrho_R(\omega)$ has three distinct roots: $\omega_{R^\mu}^1 \in (\pi, 2\pi)$, $\omega_{R^\mu}^2 \in (4\pi, \omega_{\varrho_R}^1)$, $\omega_{R^\mu}^3 \in (\omega_{\varrho_R}^1, \omega_{\varrho_R}^1)$, with $\text{sgn}(I^\mu(\omega_{R^\mu}^1)) = +1$, $\text{sgn}(I^\mu(\omega_{R^\mu}^2)) = -1$, and $\text{sgn}(I^\mu(\omega_{R^\mu}^3)) = -1$. Applying the Stépán-Hassard formula with $K = 4$, $r = 3$, and $I_\mu^{(4)}(0) = 0$ gives $\nu_+(\tilde{\Delta}_4) = 0$.

5) For $\mu > \mu_2$, strict dominance is lost, particularly over (μ_2, μ_5) , where $\nu_+(\tilde{\Delta}_4) = 2$, and over $[\mu_5, \mu_7)$, where $\nu_+(\tilde{\Delta}_4) = 4$.

6) For $\mu < 0$ corresponds to $\kappa = \kappa_- < -1$ as well as the spectral value s_- , in this case, we have $\tilde{\Delta}_4^{(4)}(0) = \frac{1}{2}\mu < 0$. Thus, by virtue of Theorem 2, $\nu_+(\tilde{\Delta}_4)$ is odd (hence nonzero). Therefore, s_- cannot be the dominant root. \square

Remark 3. 1). An approximate value of μ^* is given by $\mu^* = \mu_2 \approx 6.7704$. This is the first value of μ where the IMID property fails, as zero is no longer strictly dominant due to the emergence of a pair of purely imaginary roots $\pm i\omega_{R^\mu}^3$. Moreover, $\omega_{R^\mu}^3 \approx 15.9535$.

2). The condition on the parameter μ ensuring the dominance of the spectral value s_+ , as stated in Theorem 3, can be expressed as follows: $0 \leq \mu < \mu^* \iff -1 \leq \kappa = \kappa_+ < \frac{\mu^*}{2} - 1$, which, after some manipulation, reduces to: $-4e^{-\frac{1}{2}(a_{0,1}+5)} \leq a_{1,1} < 4\left(\frac{\mu^*}{2} - 1\right)e^{-\frac{1}{2}(a_{0,1}+5-\mu^*)}$.

Figure 1 illustrates the region $\mathcal{D}_{\text{IMID}}^+$ of validity of the IMID property, which is delimited by the two curves defined above. Note that the condition $-4e^{-\frac{1}{2}(a_{0,1}+5)} = a_{1,1}$ corresponds to $\mu = 0$ and $s_+ = s_0$, under which the MID holds [19].

IV. PRESCRIBING LINEARIZED INVERTED-PENDULUM EXPONENTIAL DECAY VIA DELAYED STATE-FEEDBACK

To demonstrate the effectiveness of the resulting PPP, consider the linear exponential stabilization of the inverted pendulum in the upright position as a standard benchmark. Assuming no friction, the dynamics of a pendulum are described by a second-order system for which the linearization admits a real-rooted characteristic polynomial. Namely, the control system

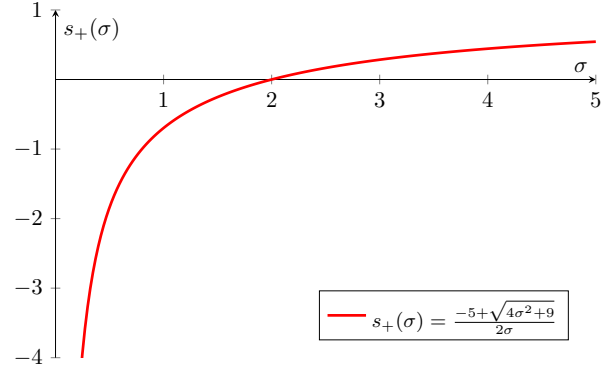


Fig. 2. Spectral abscissa at s_+ given by (13) as a function of the delay σ .

under its linearized form writes as:

$$\ddot{\xi}(t) - \frac{g}{l}\xi(t) = u(t) \quad (18)$$

where ξ represents the angular displacement from the upright position, l denotes the pendulum's length, and g is the gravitational acceleration. Several studies emphasized the positive delay effect in the SISO case by exploiting the control law $u(t) = -a_{1,0}\xi(t - \sigma) - a_{2,0}\xi(t - 2\sigma)$, including [17], [23], [24], which has highlighted the beneficial effects of delay in the control law. In particular, [17] established necessary and sufficient conditions for exponential stability, while [23], [24] proposed a center manifold approach to stabilize the nonlinear inverted pendulum model by leveraging the effects of delay. However, all these results do not give any insight on the precise decay of the closed-loop system solution. Our goal here is to benefit from the delay as a control parameter in the design of a more general delayed state-feedback control including the delayed angular velocity action:

$$u(t) = -a_{1,0}\xi(t - \sigma) - a_{1,1}\dot{\xi}(t - \sigma) - a_{2,0}\xi(t - 2\sigma) \quad (19)$$

capable of assigning the exponential decay of the solution. By considering such a class of controllers, in closed-loop of system (18) reduces to (3) with $a = 0$ and $b = -g/l$. Thanks to Theorem 3, we can set the spectral abscissa of the linearized closed-loop inverted pendulum model at the quadruple spectral value occurring at $s_+ = \frac{-5l + \sqrt{4gl\sigma^2 + 9l^2}}{2l\sigma}$ if the control parameters satisfy (\star_+) , that is $a_{1,0} = \frac{(-4s_+^2\sigma^2 - 2s_+\sigma + 2)e^{s_+\sigma}}{\sigma^2}$, $a_{1,1} = \frac{(4s_+\sigma + 6)e^{s_+\sigma}}{\sigma}$, $a_{2,0} = \frac{(s_+\sigma + 2)e^{2s_+\sigma}}{\sigma^2}$. Figure 2 illustrates the spectral abscissa s_+ , given by (13), as a function of the delay σ , for fixed parameters $g = l = 10$. Stability is ensured when $0 < \sigma < 2$, for which the spectral abscissa remains negative. By setting $\sigma = \frac{1}{2}$ and selecting control parameters as prescribed by (\star_+) in Theorem 3, Figure 3 shows the spectrum distribution of the closed-loop system. In this case, the coefficient $a_{1,1} > 0$ satisfies condition (C1) of the theorem, and the dominant root is located at $s_+ = -5 + \sqrt{10} \approx -1.83$. Finally, Figure 4 presents time-domain trajectories of the linearized system for the same parameter setting, with constant initial conditions $\xi(t) = 1$, $\dot{\xi}(t) = 5$ for all $t \in [-\sigma, 0)$. The simulations are performed in Maple, where $t \mapsto \xi(t)$ is plotted as a solid red line and $t \mapsto \dot{\xi}(t)$ as a dashed red line.

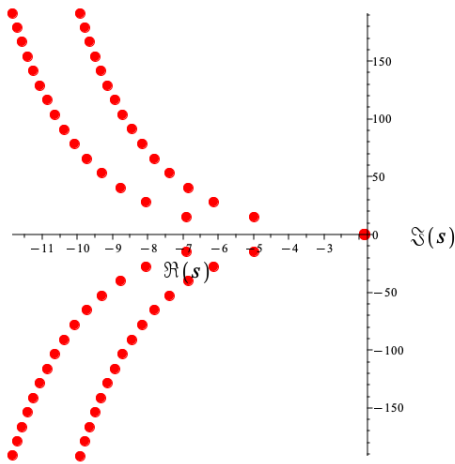


Fig. 3. Spectrum distribution of the closed-loop system (18)-(19).

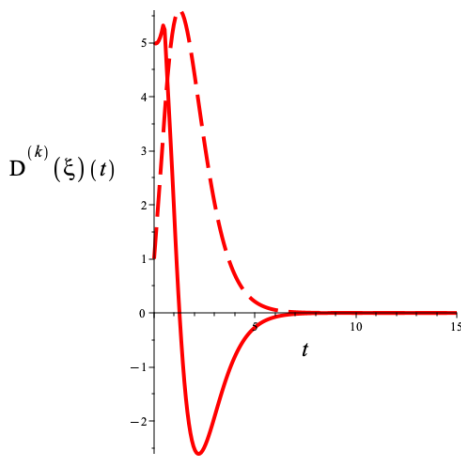


Fig. 4. Solution of the controlled linearized inverted pendulum (18)-(19).

V. CONCLUDING REMARKS

This work characterizes the multiplicity-4 IMID property for planar delay systems and shows the effectiveness of PPP in control problems with delay as *control parameter*, offering promising prospects to control MIMO systems with input delay. It is acknowledged that, similar to standard pole placement methods, PPP exhibits sensitivity to parameter variations. Nevertheless, its low computational complexity makes it particularly attractive for infinite-dimensional systems. As noted in [10], the spectral abscissa is a continuous function in system's parameters, preserving the closed-loop stability under small parameter changes. In addition, several studies have analyzed the behavior of characteristic roots under such variations. Specifically, when the delay is the varying parameter, a detailed analysis can be found in [25]. The findings will be integrated into the P3 δ software (see e.g. [26]).

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